S2S11

1. A factory produces components. Each component has a unique identity number and it is assumed that 2% of the components are faulty. On a particular day, a quality control manager wishes to take a random sample of 50 components.

   (a) Identify a sampling frame. **(1)**

   The statistic F represents the number of faulty components in the random sample of size 50.

   (b) Specify the sampling distribution of F. **(2)**

a) list of ID numbers b) $F \sim B(50, 0.02)$

2. A traffic officer monitors the rate at which vehicles pass a fixed point on a motorway. When the rate exceeds 36 vehicles per minute he must switch on some speed restrictions to improve traffic flow.

   (a) Suggest a suitable model to describe the number of vehicles passing the fixed point in a 15 s interval. **(1)**

   The traffic officer records 12 vehicles passing the fixed point in a 15 s interval.

   (b) Stating your hypotheses clearly, and using a 5% level of significance, test whether or not the traffic officer has sufficient evidence to switch on the speed restrictions. **(6)**

   (c) Using a 5% level of significance, determine the smallest number of vehicles the traffic officer must observe in a 10 s interval in order to have sufficient evidence to switch on the speed restrictions. **(3)**

a) $x \sim Po(9)$ x = Cars per 15 sec

b) $H_0 : \lambda = 9$ $P(x \geq 12) = 1 - P(x \leq 11)$
$H_1 : \lambda > 9$ $P(x > 11)$
$= 0.197$

$> 5\%$ ∴ not enough evidence to reject null hypothesis as result is not significant
∴ he does not have enough evidence to switch on speed restrictions.

c) $y \sim Po(6)$ y = Cars per 10 sec

$P(y \geq u) < 0.05$ $1 - P(y \leq u-1) < 0.05$
$P(y > u-1)$ $\Rightarrow P(y \leq u-1) > 0.95$

$P(y \leq 9) = 0.9161 < 0.95$ ∴ $u - 1 = 10$
$P(y \leq 10) = 0.9574) > 0.95$ ∴ $u = 11$

4. In a game, players select sticks at random from a box containing a large number of sticks of different lengths. The length, in cm, of a randomly chosen stick has a continuous uniform distribution over the interval [7, 10].

   A stick is selected at random from the box.

   (a) Find the probability that the stick is shorter than 9.5 cm. **(2)**

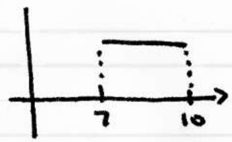
   To win a bag of sweets, a player must select 3 sticks and wins if the length of the longest stick is more than 9.5 cm.

   (b) Find the probability of winning a bag of sweets. **(2)**

   To win a soft toy, a player must select 6 sticks and wins the toy if more than four of the sticks are shorter than 7.6 cm.

   (c) Find the probability of winning a soft toy. **(4)**

a) $x \sim U[7, 10]$



$P(x < 9.5) = \frac{2.5}{3} = \frac{5}{6}$

b) ∴ player loses if all 3 < 9.5

∴ $P(\text{win}) = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$

c) $P(x < 7.6) = \frac{0.6}{3} = 0.2$

y = stick chosen is shorter than 7.6

$y \sim B(6, 0.2)$ $P(y > 4) = 1 - P(y \leq 4)$
$= 0.0016$

blank

3.

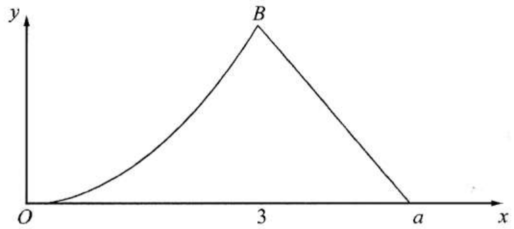


Figure 1

Figure 1 shows a sketch of the probability density function $f(x)$ of the random variable X.

For $0 \leqslant x \leqslant 3$, $f(x)$ is represented by a curve OB with equation $f(x) = kx^2$, where k is a constant.

For $3 \leqslant x \leqslant a$, where a is a constant, $f(x)$ is represented by a straight line passing through B and the point $(a, 0)$.

For all other values of x, $f(x) = 0$.

Given that the mode of X = the median of X, find

(a) the mode, **(1)**

(b) the value of k, **(4)**

(c) the value of a. **(3)**

Without calculating $E(X)$ and with reference to the skewness of the distribution

(d) state, giving your reason, whether $E(X) < 3$, $E(X) = 3$ or $E(X) > 3$. **(2)**

blank

Question 3 continued

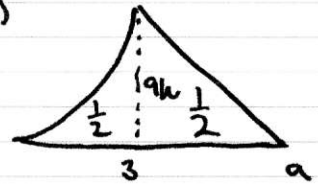
a) mode = median = 3

b) $F(Q_2) = \frac{1}{2} \therefore \int f(x)dx = \frac{1}{3}kx^3 + c$

$x=0$ $f(x)=0 \Rightarrow c=0$ $\therefore \frac{1}{3}kx^3 = \frac{1}{2}$ when $x=3$

$\Rightarrow 9k = \frac{1}{2} \therefore k = \frac{1}{18}$

c)



$\frac{9k(a-3)}{2} = \frac{1}{2}$

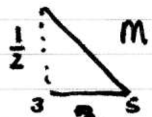
$\frac{1}{4}(a-3) = \frac{1}{2} \therefore a-3 = 2$

$\therefore \underline{a = 5}$

d) negative skew due to shape of graph

$\therefore$ mean < median $\therefore E(X) < 3$

check $E(x) = \int x f(x) dx$



$m = \frac{-\frac{1}{2}}{2} = -\frac{1}{4}$

$y = -\frac{1}{4}(x-5)$

$E(x) = \int_0^3 \frac{1}{18}x^3 dx + \int_3^5 -\frac{1}{4}(x^2 - 5x)dx$

$= \left[\frac{1}{72}x^4\right]_0^3 + \left[-\frac{1}{4}\left(\frac{1}{3}x^3 - \frac{5x^2}{2}\right)\right]_3^5$

$= \frac{9}{8} + \left[\frac{125}{24} - \frac{27}{8}\right] = \frac{71}{24} = 2.958\dot{3}$ $\therefore < 3$

bla

5. Defects occur at random in planks of wood with a constant rate of 0.5 per 10 cm length. Jim buys a plank of length 100 cm.

(a) Find the probability that Jim's plank contains at most 3 defects. **(2)**

Shivani buys 6 planks each of length 100 cm.

(b) Find the probability that fewer than 2 of Shivani's planks contain at most 3 defects. **(5)**

(c) Using a suitable approximation, estimate the probability that the total number of defects on Shivani's 6 planks is less than 18. **(6)**

a) $x \sim Po(5)$ $\quad P(x \leq 3) = \underline{0.265}$

b) $y \sim B(6, 0.265)$ $\quad P(y<2) = P(y=0)+P(y=1)$

$= 0.735^6 + 6 \times 0.735^5 \times 0.265 = \underline{0.499}$

c) $d \sim Po(30) \approx d \sim N(30, 30)$

$P(d<18)$ $P(d \leq 17)$ cc $P(d<17.5) \approx P\left(Z < \frac{17.5-30}{\sqrt{30}}\right)$

$\approx P(Z<-2.28)$

$= 1-\Phi(2.28) = \underline{0.0113}$

6. A shopkeeper knows, from past records, that 15% of customers buy an item from the display next to the till. After a refurbishment of the shop, he takes a random sample of 30 customers and finds that only 1 customer has bought an item from the display next to the till.

(a) Stating your hypotheses clearly, and using a 5% level of significance, test whether or not there has been a change in the proportion of customers buying an item from the display next to the till. **(6)**

During the refurbishment a new sandwich display was installed. Before the refurbishment 20% of customers bought sandwiches. The shopkeeper claims that the proportion of customers buying sandwiches has now increased. He selects a random sample of 120 customers and finds that 31 of them have bought sandwiches.

(b) Using a suitable approximation and stating your hypotheses clearly, test the shopkeeper's claim. Use a 10% level of significance. **(8)**

a) $x \sim B(30, 0.15)$ $\quad P(x \leq 1) = 0.048 \; (<0.05)$

$H_0: P = 0.15$
$H_1: P < 0.15$

$\therefore$ enough evidence to reject null hypothesis as result is significant

$\therefore$ enough evidence to suggest proportion of customers buying item next to till has reduced

b) $y \sim B(120, 0.2)$ $\quad np = \mu = 24$

$np(1-p) = \sigma^2 = 24 \times 0.8 = 19.2$

$\approx N \sim N(24, 19.2)$

$P(x \geq 31)$ $P(x>30)$ cc $P(x>30.5) \approx P\left(Z > \frac{30.5-24}{\sqrt{19.2}}\right)$

$\approx P(Z>1.48) = 1-\Phi(1.48) = 0.0694 \; (<0.1)$

$\therefore H_0: P=0.2$
$H_1: P>0.2$

enough evidence to reject null hypothesis as result is significant

$\therefore$ claim is correct.

7. The continuous random variable X has probability density function given by

$$f(x)=\begin{cases}\frac{3}{32}(x-1)(5-x) & 1\leqslant x\leqslant 5\\ 0 & \text{otherwise}\end{cases}$$

(a) Sketch $f(x)$ showing clearly the points where it meets the x-axis. **(2)**

(b) Write down the value of the mean, μ, of X. **(1)**

(c) Show that $E(X^2)=9.8$ **(4)**

(d) Find the standard deviation, σ, of X. **(2)**

The cumulative distribution function of X is given by

$$F(x)=\begin{cases}0 & x<1\\ \frac{1}{32}\left(a-15x+9x^2-x^3\right) & 1\leqslant x\leqslant 5\\ 1 & x>5\end{cases}$$

where a is a constant.

(e) Find the value of a. **(2)**

(f) Show that the lower quartile of X, q_1, lies between 2.29 and 2.31 **(3)**

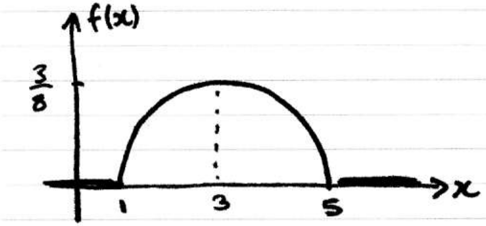
(g) Hence find the upper quartile of X, giving your answer to 1 decimal place. **(1)**

(h) Find, to 2 decimal places, the value of k so that

$$P(\mu-k\sigma<X<\mu+k\sigma)=0.5$$

(2)

Question 7 continued



b) $E(x) = 3$

c) $E(x^2) = \int_1^5 x^2 f(x)dx = \int_1^5 \frac{3}{32}x^2(-x^2+6x-5)dx$

$= \frac{3}{32}\int_1^5 -x^4+6x^3-5x^2\,dx = \frac{3}{32}\left[-\frac{1}{5}x^5+\frac{3}{2}x^4-\frac{5}{3}x^3\right]_1^5$

$= \frac{3}{32}\left[\frac{625}{6}-\left(\frac{-11}{30}\right)\right] = \frac{49}{5} = 9.8$

d) $V(X) = E(X^2) - E(X)^2 = 9.8 - 3^2 = 0.8$

$\sigma = \sqrt{V(x)} = 0.894$

e) $F(1) = 0 \Rightarrow \frac{1}{32}(a-15+9-1) = 0 \quad \therefore a = 7$

f) $F(Q_1) = 0.25 \qquad F(2.29) = 0.245 < 0.25$

$F(2.31) = 0.252 > 0.25$

$\therefore 2.29 < Q_2 < 2.31$

g) $F(Q_3) = 0.75 \Rightarrow \frac{1}{32}(7-15x+9x^2-x^3) = \frac{3}{4}$

$24 = 7 - 15x + 9x^2 - x^3 \quad \therefore x^3 - 9x^2 + 15x + 17 = 0$

$\therefore x = 3.69 \quad \therefore Q_3 = 3.7$

h) $P(\mu - k\sigma < X < \mu + k\sigma) = 0.5 \qquad \Rightarrow 3 + 0.894k = 3.69$

$\Rightarrow P(Q_1 < X < Q_3) = 0.5 \qquad \therefore k = 0.77$