S2 W13

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1. (a) Write down the conditions under which the Poisson distribution can be used as an approximation to the binomial distribution.
(2)

The probability of any one letter being delivered to the wrong house is 0.01
On a randomly selected day Peter delivers 1000 letters.

(b) Using a Poisson approximation, find the probability that Peter delivers at least 4 letters to the wrong house.

Give your answer to 4 decimal places.
(3)

a) large n, small p $\quad np \leq 10$

b) $np = 10 \quad X \sim Po(10) \quad P(X \geq 4) \quad P(X > 3)$

$= 1 - P(X \leq 3) = 1 - 0.0103 = 0.9897$

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2. In a village, power cuts occur randomly at a rate of 3 per year.

(a) Find the probability that in any given year there will be

(i) exactly 7 power cuts,

(ii) at least 4 power cuts.
(5)

(b) Use a suitable approximation to find the probability that in the next 10 years the number of power cuts will be less than 20
(6)

a) $X \sim Po(3) \quad P(X = 7) = \dfrac{e^{-3} \times 3^7}{7!} = 0.0216$

b) $P(X \geq 4)$ $P(X > 3)$ $\Rightarrow 1 - P(X \leq 3) = 1 - 0.6472$

$= 0.3528$

c) 10 years $\mu = 30$ $\quad P(Y < 20)$

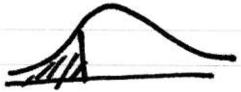
Y = power cuts in 10 years

$Y \sim Po(30) \simeq Y \sim N(30, 30)$

$P(Y < 20) \quad P(Y \leq 19) \approx P(Y < 19.5) \simeq P\left(Z < \dfrac{19.5 - 30}{\sqrt{30}}\right)$

$\simeq P(Z < -1.92) = 1 - \Phi(1.92)$

$= 1 - 0.9726 = 0.0274$



-1.92

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3. A random variable X has the distribution $B(12, p)$.

(a) Given that $p = 0.25$ find

(i) $P(X < 5)$

(ii) $P(X \geqslant 7)$

(3)

(b) Given that $P(X = 0) = 0.05$, find the value of p to 3 decimal places.

(3)

(c) Given that the variance of X is 1.92, find the possible values of p.

(4)

a) $X \sim B(12, 0.25)$

i) $P(X<5) = P(X \leq 4) = 0.8424$

ii) $P(X \geq 7)$ $P(X > 6) = 1 - P(X \leq 6) = 0.0143$

b) $P(X=0) = 0.05 \Rightarrow p^{12} = 0.05$

$\Rightarrow p = \sqrt[12]{0.05}$

$= 0.779$

c) $V(X) = np(1-p)$ $\quad 12p - 12p^2 = 1.92$

$12p^2 - 12p + 1.92 = 0$

$p = \dfrac{12 \pm \sqrt{12^2 - 4(12)(1.92)}}{24} = 0.2, 0.8$

4. The continuous random variable X is uniformly distributed over the interval $[-4, 6]$.

(a) Write down the mean of X.

(1)

(b) Find $P(X \leqslant 2.4)$

(2)

(c) Find $P(-3 < X - 5 < 3)$

(2)

The continuous random variable Y is uniformly distributed over the interval $[a, 4a]$.

(d) Use integration to show that $E(Y^2) = 7a^2$

(4)

(e) Find $\text{Var}(Y)$.

(2)

(f) Given that $P(X < \frac{8}{3}) = P(Y < \frac{8}{3})$, find the value of a.

(3)

a) $E(X) = \dfrac{a+b}{2} = 1$ $\qquad$ b) $P(X \leq 2.4) = \dfrac{6.4}{10} = 0.64$

c) $P(-3 < X - 5 < 3) = P(2 < X < 8)$

$= P(2 < X < 6)$ 6 is max

$= \dfrac{4}{10} = 0.4$

d) $E(y) = \displaystyle\int_a^{4a} \frac{y^2}{3a}\,dy = \left[\frac{y^3}{9a}\right]_a^{4a} = \frac{64a^3 - a^3}{9a}$

$\therefore E(y) = 7a^2$

e) $V(y) = E(y^2) - E(y)^2 = 7a^2 - (\frac{5}{2}a)^2 = \frac{3}{4}a^2$

f) $P(X < \frac{8}{3}) = \dfrac{2\frac{2}{3} + 4}{10} = \dfrac{2}{3}$

$P(y < \frac{8}{3}) = \dfrac{\frac{8}{3} - a}{3a} = \dfrac{2}{3} \Rightarrow \frac{8}{3} - a = 2a$

$3a = \frac{8}{3} \quad \therefore a = \frac{8}{9}$

5. The continuous random variable T is used to model the number of days, t, a mosquito survives after hatching.

The probability that the mosquito survives for more than t days is

$$\frac{225}{(t+15)^2}, \quad t \geqslant 0$$

(a) Show that the cumulative distribution function of T is given by

$$\mathrm{F}(t) = \begin{cases} 1 - \dfrac{225}{(t+15)^2} & t \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

(1)

(b) Find the probability that a randomly selected mosquito will die within 3 days of hatching.

(2)

(c) Given that a mosquito survives for 3 days, find the probability that it will survive for at least 5 more days.

(3)

A large number of mosquitoes hatch on the same day.

(d) Find the number of days after which only 10% of these mosquitoes are expected to survive.

(4)

a) $P(T>t) = \dfrac{225}{(t+15)^2} \Rightarrow P(T \leq t) = 1 - P(T>t)$

$= 1 - \dfrac{225}{(t+15)^2}$

$F(t) = P(T \leq t) = \begin{cases} 1 - \dfrac{225}{(t+15)^2} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

b) $P(T \leq 3) = 1 - \dfrac{225}{(t+15)^2} = 1 - \dfrac{225}{18^2} = \dfrac{11}{36}$

c) $P(T>3) = \dfrac{25}{36}$ $\quad P(T>8 \mid T>3) = \dfrac{P(T>8)}{P(T>3)}$

$= \dfrac{\frac{225}{23^2}}{\frac{225}{18^2}} = \left(\dfrac{18}{23}\right)^2 = 0.612$

2

d) $\dfrac{225}{(t+15)^2} = 0.1$

$\Rightarrow (t+15)^2 = 2250 \Rightarrow t+15 = 15\sqrt{10}$

$\therefore t = -15 + 15\sqrt{10} \quad \therefore t = 32.4$

6. (a) Explain what you understand by a hypothesis.

(1)

(b) Explain what you understand by a critical region.

(2)

Mrs George claims that 45% of voters would vote for her.

In an opinion poll of 20 randomly selected voters it was found that 5 would vote for her.

(c) Test at the 5% level of significance whether or not the opinion poll provides evidence to support Mrs George's claim.

(4)

In a second opinion poll of n randomly selected people it was found that no one would vote for Mrs George.

(d) Using a 1% level of significance, find the smallest value of n for which the hypothesis $\mathrm{H}_0 : p = 0.45$ will be rejected in favour of $\mathrm{H}_1 : p < 0.45$

(3)

a) Statement about a population parameter

b) range of values in which the null hypothesis is rejected as the test would be significant

c) $H_0 : p = 0.45$ $\quad P(X \leq L) < 0.05$

$H_1 : p < 0.45$ $\quad P(X \leq 4) = 0.0189$ ✓

$X \sim B(20, 0.45)$ $\quad P(X \leq 5) = 0.0553$

$\therefore$ CR $X \leq 4$

5 is not in the CR $\therefore$

$\therefore$ not enough evidence to reject null hypothesis as test is not significant.

$\therefore$ evidence to support his claim.

d) $Y \sim B(n, 0.45)$

$n=7 \quad B(7, 0.45) \Rightarrow P(X=0) = 0.0152 > 1\%$

$n=8 \quad B(8, 0.45) \Rightarrow P(X=0) = 0.0084 < 1\%$

$\therefore n = 8$

7. The continuous random variable X has the following probability density function

$$f(x) = \begin{cases} a + bx & 0 \leqslant x \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$$

where a and b are constants.

(a) Show that $10a + 25b = 2$ **(4)**

Given that $E(X) = \frac{35}{12}$

(b) find a second equation in a and b, **(3)**

(c) hence find the value of a and the value of b. **(3)**

(d) Find, to 3 significant figures, the median of X. **(3)**

(e) Comment on the skewness. Give a reason for your answer. **(2)**

a) $\int f(x)dx = 1 \Rightarrow \int_0^5 a + bx \, dx = 1$

$\Rightarrow \left[ax + \frac{b}{2}x^2\right]_0^5 = 5a + \frac{25}{2}b = 1 \quad \therefore 10a + 25b = 2$ #

b) $E(x) = \int x f(x) dx = \int_0^5 ax + bx^2 dx = \left[\frac{1}{2}ax^2 + \frac{bx^3}{3}\right]_0^5$

$\Rightarrow \frac{25}{2}a + \frac{125}{3}b = \frac{35}{12}$ $\times 12$ $150a + 500b = 35$

$30a + 100b = 7$

$30a + 75b = 6$ $-$

$25b = 1 \quad b = \frac{1}{25}$

$\Rightarrow 10a + 1 = 2 \quad a = \frac{1}{10}$

d) $F(x) = \int_0^x a + bt \, dt = \left[at + \frac{b}{2}t^2\right]_0^x = ax + \frac{b}{2}x^2$

$\therefore F(x) = \frac{1}{10}x + \frac{1}{50}x^2$

$F(Q_2) = 0.5 \Rightarrow \frac{1}{10}x + \frac{1}{50}x^2 = \frac{1}{2}$ $\times 50$ $x^2 + 5x - 25 = 0$

$x = \frac{-5 \pm \sqrt{5^2 + 4(25)}}{2} \quad \therefore Q_2 = 3.09$

Mean 2.92 < Median 3.09 $\therefore$ negative skew