FP3 June 2013 M. A. Kprime2

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1. A hyperbola H has equation

$$\frac{x^2}{a^2} - \frac{y^2}{25} = 1, \qquad \text{where } a \text{ is a positive constant.}$$

The foci of H are at the points with coordinates (13, 0) and (–13, 0).

Find

(a) the value of the constant a, **(3)**

(b) the equations of the directrices of H. **(3)**

1(a). Foci $(ae, 0) \Leftrightarrow (13, 0)$

$\Rightarrow ae = 13$

$\Rightarrow e = \frac{13}{a}$

Eccentricity: $b^2 = a^2(e^2 - 1)$

$\therefore 25 = a^2\left(\frac{169}{a^2} - 1\right)$

$\therefore 25 = 169 - a^2$

$\therefore a^2 = 144 \Rightarrow a = 12$

(b) ~~$x = \pm 12$~~ $e = \frac{13}{12}$

$\therefore x = \pm \frac{144}{13}$

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2. (a) Find

$$\int \frac{1}{\sqrt{(4x^2+9)}}\,dx$$

(2)

(b) Use your answer to part (a) to find the exact value of

$$\int_{-3}^{3} \frac{1}{\sqrt{(4x^2+9)}}\,dx$$

giving your answer in the form $k\ln(a + b\sqrt{5})$, where a and b are integers and k is a constant.

(3)

2(a). $\int \frac{1}{\sqrt{4x^2+9}}\,\partial x = \int \frac{1}{\sqrt{(2x)^2+9}}\,\partial x$

$= \frac{1}{2}\text{arsinh}\left(\frac{2x}{3}\right) + C$

7.253...

~~(b)~~ $\cancel{\int \frac{1}{\sqrt{4(x^2+\frac{9}{4})}}\,\partial x = \frac{1}{2}\int \frac{1}{\sqrt{x^2+9/4}}\,\partial x}$

$\cancel{= \frac{1}{2}\left[= \frac{1}{2}\text{arsinh}\left(\frac{2x}{3}\right) + C\right.}$

(b) $\left[\frac{1}{2}\text{arsinh}\left(\frac{2x}{3}\right)\right]_{-3}^{3} = \frac{1}{2}\text{arsinh}\,2 - \frac{1}{2}\text{arsinh}(-2)$

$= \frac{1}{2}\left(\ln(2+\sqrt{5}) - \ln(-2+\sqrt{5})\right)$

$= \frac{1}{2}\ln(9+4\sqrt{5})$

3. The curve with parametric equations

$$x = \cosh 2\theta, \quad y = 4 \sinh \theta, \quad 0 \leqslant \theta \leqslant 1$$

is rotated through 2π radians about the x-axis.

Show that the area of the surface generated is $\lambda(\cosh^3 \alpha - 1)$, where $\alpha = 1$ and λ is a constant to be found.

(7)

3. S. Area $= 2\pi \int_0^1 y \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2} \, d\theta$

$x = \cosh 2\theta \Rightarrow \frac{\partial x}{\partial \theta} = 2\sinh 2\theta$

$y = 4\sinh\theta \Rightarrow \frac{\partial y}{\partial \theta} = 4\cosh\theta$

$(\sinh 2\theta)^2 = 4\sinh^2\theta\cosh^2\theta$

$\therefore$ Area $= 2\pi \int_0^1 4\sinh\theta \sqrt{4\sinh^2 2\theta + 16\cosh^2\theta} \, \partial\theta$

$c^2 - s^2 = 1$

$\cosh^2\theta = 1 + \sinh^2\theta$

$= 2\pi \int_0^1 4\sinh\theta \sqrt{16\sinh^2\theta\cosh^2\theta + 16\cosh^2\theta} \, \partial\theta$

$= 2\pi \int_0^1 4\sinh\theta \sqrt{16\cosh^2\theta(\sinh^2\theta + 1)} \, \partial\theta$

$= 2\pi \int_0^1 4\sinh\theta \sqrt{16\cosh^4\theta} \, \partial\theta$

$$= 8\pi \int_0^1 \sinh\theta \cdot 4\cosh^2\theta \; \partial\theta$$

$$= 32\pi \int_0^1 \sinh\theta \cosh^2\theta \; \partial\theta \qquad \int f'(x)[f(x)]^n \partial x$$

$$= 32\pi \left[\frac{\cosh^3\theta}{3} \right]_0^1$$

$$= 32\pi \left(\frac{\cosh^3(1)}{3} - \frac{\cosh^3(0)}{3} \right)$$

$$= 32\pi \left(\frac{1}{3}\cosh^3(1) - \frac{1}{3} \right)$$

$$= \frac{32}{3}\pi \left(\cosh^3(1) - 1 \right)$$

$$\lambda = \frac{32}{3}\pi$$

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4.

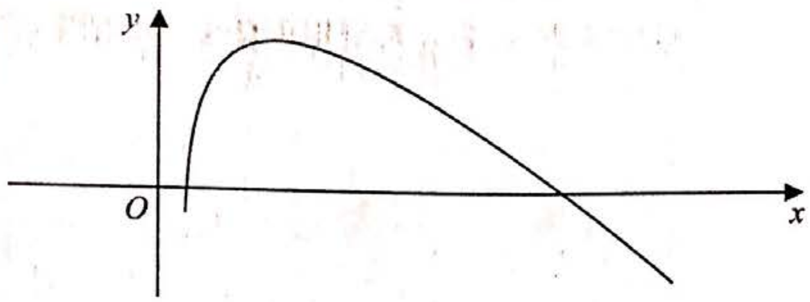

Figure 1

Figure 1 shows part of the curve with equation

$$y = 40 \operatorname{arcosh} x - 9x, \qquad x \geqslant 1$$

Use calculus to find the exact coordinates of the turning point of the curve, giving your answer in the form $\left(\frac{p}{q}, r\ln 3 + s\right)$, where p, q, r and s are integers.

(7)

4. $y = 40 \operatorname{arcosh} x - 9x$

$\therefore \frac{dy}{dx} = \frac{40}{\sqrt{x^2-1}} - 9$

$\frac{dy}{dx} = 0 \Rightarrow \frac{40}{\sqrt{x^2-1}} - 9 = 0$

$\therefore 40 = 9\sqrt{x^2-1}$

$\Rightarrow 40^2 = \left(9\sqrt{x^2-1}\right)^2$

$\therefore 1600 = 81(x^2-1)$

$\therefore \frac{1600}{81} = x^2 - 1 \Rightarrow x = +\frac{41}{9}$

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$x = \frac{41}{9}$

$\Rightarrow \quad y = 40\operatorname{arcosh}\frac{41}{9} - 41$

$= 40\left(\ln\left(\frac{41}{9} + \sqrt{\frac{1600}{81}}\right)\right) - 41$

$= 40\ln\left(\frac{41}{9} + \frac{40}{9}\right) - 41$

$= 40\ln(9) - 41$

$= 40\ln(3^2) - 41$

$= 80\ln(3) - 41$

$\therefore$ Turning point: $\left(\frac{41}{9}, \ 80\ln(3) - 41\right)$

$p = 41 \quad q = 9 \quad r = 80$

$s = -41$

Q4

(Total 7 marks)

5. The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & a \\ 2 & b & c \\ -1 & 0 & 1 \end{pmatrix}, \text{ where } a, b \text{ and } c \text{ are constants.}$$

(a) Given that $\mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{k}$ are two of the eigenvectors of **M**,

find

(i) the values of a, b and c,

(ii) the eigenvalues which correspond to the two given eigenvectors.

(8)

(b) The matrix **P** is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & d \\ -1 & 0 & 1 \end{pmatrix}, \text{ where } d \text{ is constant, } d \neq -1$$

Find

(i) the determinant of **P** in terms of d,

(ii) the matrix $\mathbf{P}^{-1}$ in terms of d.

(5)

5(a) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are e.vectors

$Mx = \lambda x \Rightarrow \begin{pmatrix} 1 & 1 & a \\ 2 & b & c \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ \lambda \end{pmatrix}$

$\therefore \begin{pmatrix} 1+a \\ b+c \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ \lambda \end{pmatrix} \Rightarrow \begin{aligned} a &= -1 \\ \lambda &= 1 \\ b+c &= 1 \end{aligned}$

Question 5 continued

$$\& \begin{pmatrix} 1 & 1 & -1 \\ 2 & b & c \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ -\lambda \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2 \\ 2-c \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ -\lambda \end{pmatrix} \Rightarrow \begin{matrix} \lambda = 2 \\ c = 2 \end{matrix}$$

$$\therefore \quad b + 2 = 1 \Rightarrow \quad b = -1$$

(i) $a = -1$
$b = -1$
$c = 2$

(ii) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ has e. value $\lambda = 1$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ has e. value $\lambda = 2$

(b)(i) $\det(P) = \begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 2 & d \\ -1 & 1 \end{vmatrix} + 0$

$= 1 - (2+d) = 1-2-d$

$= -(1+d)$

(ii)

$$C = \begin{pmatrix} +(1) & -(2+d) & +(1) \\ -(1) & +(1) & -(1) \\ +(d) & -(d) & +(-1) \end{pmatrix}$$

$$\therefore C^T = \begin{pmatrix} 1 & -1 & d \\ -2-d & 1 & -d \\ 1 & -1 & -1 \end{pmatrix}$$

$$\therefore P^{-1} = \frac{-1}{d+1} \begin{pmatrix} 1 & -1 & d \\ -2-d & 1 & -d \\ 1 & -1 & -1 \end{pmatrix}$$

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6. Given that

$$I_n = \int_0^4 x^n \sqrt{(16 - x^2)}\,dx, \quad n \geqslant 0,$$

(a) prove that, for $n \geqslant 2$,

$$(n + 2)I_n = 16(n - 1)I_{n-2}$$

(6)

(b) Hence, showing each step of your working, find the exact value of I_5

(5)

(a). $I_n = \int_0^4 x^n \sqrt{16-x^2}\ \partial x$

$= -\frac{1}{2} \int_0^4 x^{n-1} \cdot -2x\sqrt{16-x^2}\ \partial x$

Let $u = x^{n-1}$ $\quad u' = (n-1)x^{n-2}$

$v' = -2x\sqrt{16-x^2} \Rightarrow \quad v = \frac{2}{3}(16-x^2)^{3/2}$

$\therefore -2I_n = \left[\frac{2x^{n-1}}{3}(16-x^2)^{3/2}\right]_0^4 - \frac{2}{3}(n-1)\int_0^4 x^{n-2}(16-x^2)^{3/2}\,\partial x$

$\therefore -2I_n = -\frac{2}{3}(n-1)\int_0^4 x^{n-2}\left(\sqrt{16-x^2}\cdot 16-x^2\right)\partial x$

$\therefore \quad I_n = \frac{n-1}{3}\int_0^4 16x^{n-2}\sqrt{16-x^2} - x^n\sqrt{16-x^2}\ \partial x$

$$\therefore I_n = \frac{n-1}{3}\left(16\int_0^4 x^{n-2}\sqrt{16-x^2}\,\partial x - \int_0^4 x^n\sqrt{16-x^2}\,\partial x\right)$$

$$\therefore I_n = \frac{n-1}{3}\left(16 I_{n-2} - I_n\right)$$

$$\therefore I_n = \frac{16}{3}(n-1) I_{n-2} - \frac{n-1}{3} I_n$$

$$\therefore \left(1+\frac{n-1}{3}\right) I_n = \frac{16}{3}(n-1) I_{n-2}$$

$$\therefore (n+2) I_n = 16(n-1) I_{n-2}$$

as required.

Question 6 continued $(n+2) I_n = 16(n-1) I_{n-2}$

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(b)

let $n=5$

$\Rightarrow \quad 7 I_5 = 64 I_3$

now $n=3$

$\Rightarrow \quad 5 I_3 = 32 I_1 \Rightarrow I_3 = \frac{32}{5} I_1$

$\therefore \quad 7 I_5 = \frac{2048}{5} I_1$

$\Rightarrow \quad I_5 = \frac{2048}{35} I_1$

$\Rightarrow I_5 = \frac{2048}{35} \int_0^4 x\sqrt{16-x^2}\, dx$

$= \frac{2048}{35} \cdot -\frac{1}{2} \int_0^4 -2x \left(16-x^2\right)^{1/2} dx$

$= -\frac{1024}{35} \left[\frac{2(16-x^2)^{3/2}}{3} \right]_0^4$

$= -\frac{1024}{35} \left(0 - \frac{128}{3} \right) = \frac{1024}{35} \times \frac{128}{3} = \frac{131072}{105}$

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7. The ellipse E has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b > 0$$

The line l is a normal to E at a point $P(a\cos\theta, b\sin\theta)$, $0 < \theta < \frac{\pi}{2}$

(a) Using calculus, show that an equation for l is

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

(5)

The line l meets the x-axis at A and the y-axis at B.

(b) Show that the area of the triangle OAB, where O is the origin, may be written as $k\sin 2\theta$, giving the value of the constant k in terms of a and b.

(4)

(c) Find, in terms of a and b, the exact coordinates of the point P, for which the area of the triangle OAB is a maximum.

(3)

7(a). $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{\partial y}{\partial x} = \frac{\partial y/\partial \theta}{\partial x/\partial \theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

$$\therefore \quad m_N = -\frac{1}{-\frac{b}{a}\frac{\cos\theta}{\sin\theta}} = \frac{a}{b}\tan\theta$$

$$\therefore \quad y - b\sin\theta = \frac{a}{b}\tan\theta(x - a\cos\theta)$$

$$\therefore \quad y - b\sin\theta = \frac{a}{b}\frac{\sin\theta}{\cos\theta}x - \frac{a^2}{b}\sin\theta$$

$$\therefore \quad by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$

$$\therefore \quad ax\sin\theta - by\cos\theta = a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta$$

Question 7 continued

$\therefore \quad ax\sin\theta - by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$

as required.

(b) at A, $y=0 \Rightarrow \quad ax\sin\theta = (a^2-b^2)\sin\theta\cos\theta$

$\Rightarrow \quad x = \dfrac{a^2-b^2}{a}\cos\theta$

~~$A\left(\dfrac{a^2-b^2}{a}\cos\theta, 0\right)$~~

$A\left(\dfrac{a^2-b^2}{a}\cos\theta \,,\, 0\right)$

At B, $x=0 \Rightarrow \quad -by\cos\theta = (a^2-b^2)\sin\theta\cos\theta$

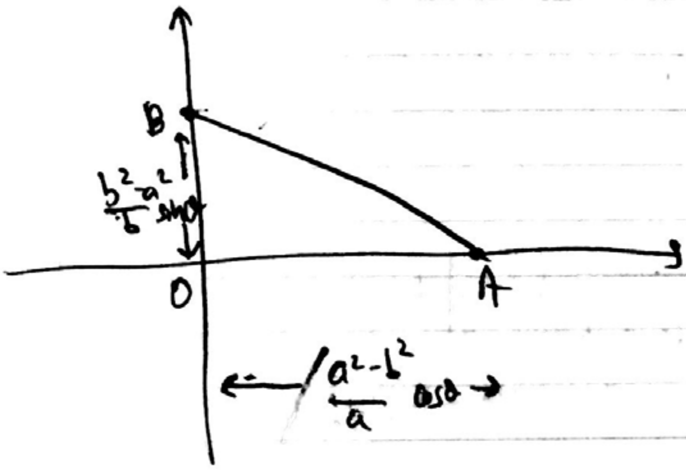
$\div -b\cos\theta$

$\therefore$

$y = \dfrac{b^2-a^2}{b}\sin\theta$

$\therefore B\left(0, \dfrac{b^2-a^2}{b}\sin\theta\right)$

Question 7 continued



$\sin\theta\cos\theta \equiv \frac{1}{2}\sin 2\theta$

$$\therefore \text{Area} = \left| \frac{1}{2} \times \frac{a^2-b^2}{a}\cos\theta \times \frac{b^2-a^2}{b}\sin\theta \right|$$

$$= \left| \frac{1}{2} \cdot \frac{(a^2-b^2)(b^2-a^2)}{ab} \cdot \frac{1}{2}\sin 2\theta \right|$$

$$= \left| \frac{(a^2-b^2)(b^2-a^2)}{4ab}\sin 2\theta \right|$$

$$= \left| -\frac{(a^2-b^2)^2}{4ab}\sin 2\theta \right|$$

$$\text{Area} = \frac{(a^2-b^2)^2}{4ab}\sin 2\theta$$

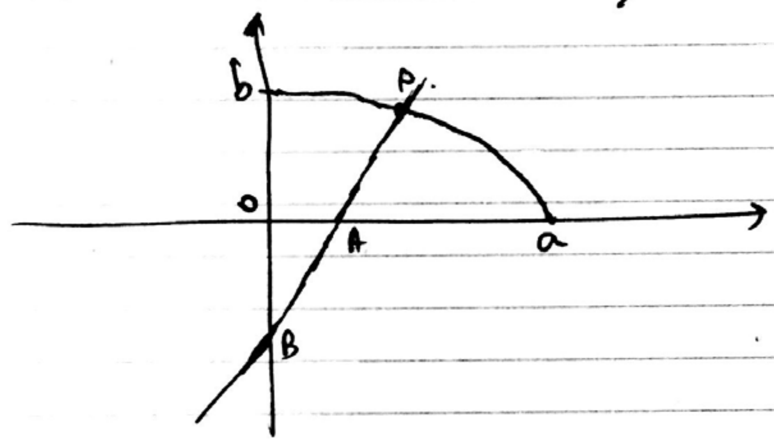
$$k = \frac{+(a^2-b^2)^2}{4ab}$$

(c) ~~Ar~~ let Area $= A$

$$A = \frac{(a^2-b^2)^2}{4ab}\sin 2\theta$$

Area is maximum when $\sin 2\theta = 1$

$$\Rightarrow \quad 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$



$\therefore$ Area is max for $\theta = \frac{\pi}{4}$

$\rightarrow P\left(a\cos\frac{\pi}{4},\ b\sin\frac{\pi}{4}\right)$

$\therefore P:\left(\frac{a}{\sqrt{2}},\ \frac{b}{\sqrt{2}}\right)$

Q7

(Total 12 marks)

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8. The plane Π_1 has vector equation

$$\mathbf{r}.(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 5$$

(a) Find the perpendicular distance from the point (6, 2, 12) to the plane Π_1

(3)

The plane Π_2 has vector equation

$$\mathbf{r} = \lambda(2\mathbf{i} + \mathbf{j} + 5\mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} - 2\mathbf{k}),$$ where λ and μ are scalar parameters.

(b) Find the acute angle between Π_1 and Π_2 giving your answer to the nearest degree.

(5)

(c) Find an equation of the line of intersection of the two planes in the form $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

(6)

8(a). $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = 5$

$\Rightarrow \quad 3x - 4y + 2z = 5$

$\Rightarrow \quad 3x - 4y + 2z - 5 = 0$

$\begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \begin{pmatrix} 6 \\ 2 \\ 12 \end{pmatrix}$

Use formula in booklet pg 10.

$$d = \frac{\left| 3(6) - 4(2) + 2(12) - 5 \right|}{\sqrt{3^2 + 4^2 + 2^2}}$$

$$= \frac{29}{\sqrt{29}} = \sqrt{29}$$

(b) $\underline{\Pi_2:}$

$$\underset{\sim}{n} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

$$= \times \begin{array}{cccccc} 2 & 1 & 5 & 2 & 1 & 5 \\ 1 & -1 & -2 & 1 & -1 & -2 \end{array}$$

$$= \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix}$$

$$\therefore \cos\theta = \left| \frac{\begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}}{\sqrt{3^2+9^2+3^2}\ \sqrt{3^2+4^2+2^2}} \right|$$

$$= \left| \frac{-33}{3\sqrt{319}} \right| = \left| -\frac{\sqrt{319}}{29} \right|$$

$$\therefore \cos\theta = \frac{\sqrt{319}}{29} \Rightarrow \theta = 52^\circ \text{ (nearest degree)}$$

(c) $\Pi_2 :$ $\underline{r} \cdot \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix}$

$\therefore \underline{r} \cdot \begin{pmatrix} 3 \\ 9 \\ -3 \end{pmatrix} = 0$

$\therefore \quad 3x + 9y - 3z = 0$

$\therefore \quad x + 3y - z = 0 \quad \Rightarrow \quad x = z - 3y$

& $\Pi_1 \Rightarrow \quad 3x - 4y + 2z = 5$

$\therefore 3z - 9y - 4y + 2z = 5$

$\therefore \quad 5z - 13y = 5$

$\Rightarrow 5z = 5 + 13y$

$\therefore z = 1 + \frac{13}{5}y$

~~$\therefore x = z - 3\left(1 + \frac{13}{5}y\right) = z$~~

$x = 1 + \frac{13}{5}y - 3y = 1 - \frac{2}{5}y$

$y = y$

$z = 1 + \frac{13}{5}y$

Leave blank

$\Rightarrow$ line of intersection has eqn

Let ~~y = λ~~ $y = t$

$$\underline{r} = \begin{pmatrix} 1 - \frac{2}{5}t \\ t \\ 1 + \frac{13}{5}t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2/5 \\ 1 \\ 13/5 \end{pmatrix}$$

$$(\underline{r} - \underline{a}) \times \underline{b} = 0$$

$$\therefore \underline{r} \times \begin{pmatrix} -2/5 \\ 1 \\ 13/5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -2/5 \\ 1 \\ 13/5 \end{pmatrix}$$

$$\begin{matrix} 1 & 0 & 1 & 1 & 0 & 1 \\ -\frac{2}{5} & 1 & \frac{13}{5} & -\frac{2}{5} & 1 & \frac{13}{5} \end{matrix}$$

$$\begin{pmatrix} -1 \\ \cancel{-3} \\ 1 \end{pmatrix}$$

$$\Rightarrow \underline{r} \times \begin{pmatrix} -2/5 \\ 1 \\ 13/5 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

$$r \times \frac{1}{5} \begin{pmatrix} -2 \\ 5 \\ 13 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

$$\therefore r \times \begin{pmatrix} -2 \\ 5 \\ 13 \end{pmatrix} = \begin{pmatrix} -5 \\ -15 \\ 5 \end{pmatrix}$$