FP2 S14 UK

1. (a) Express $\dfrac{2}{(r+2)(r+4)}$ in partial fractions.

(1)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \frac{n(7n+25)}{12(n+3)(n+4)}$$

(5)

$\dfrac{A}{r+2} + \dfrac{B}{r+4} \Rightarrow \quad 2 = A(r+4) + B(r+2)$

$r = -2 \Rightarrow \underline{A=1} \quad r = -4 \Rightarrow \underline{B=-1}$

$\therefore \quad = \dfrac{1}{r+2} - \dfrac{1}{r+4}$

b) $\displaystyle\sum_{1}^{n} \frac{2}{(r+2)(r+4)} = \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{4} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \dots + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+3}\right) + \left(\cancel{\frac{1}{n+2}} - \frac{1}{n+4}\right)$

$= \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{-1}{n+3} - \dfrac{1}{n+4} \quad = \dfrac{7}{12} - \dfrac{1}{n+3} - \dfrac{1}{n+4}$

$= \dfrac{7(n+3)(n+4) - 12(n+4) - 12(n+3)}{12(n+3)(n+4)}$

$= \dfrac{7n^2 + 49n + 84 - 12n - 48 - 12n - 36}{12(n+3)(n+4)}$

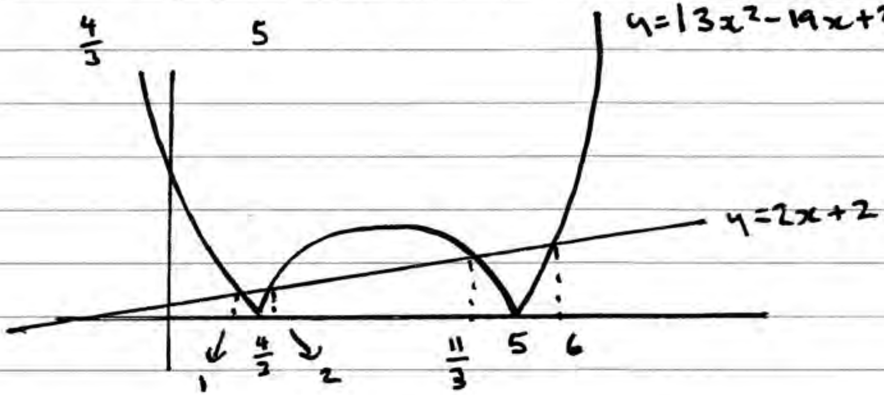
$= \dfrac{7n^2 + 25n}{12(n+3)(n+4)} = \dfrac{n(7n+25)}{12(n+3)(n+4)}$

2. Use algebra to find the set of values of x for which

$$|3x^2 - 19x + 20| < 2x + 2$$

(6)

$|(3x-4)(x-5)| < 2x+2$



$3x^2 - 19x + 20 = 2x + 2$

$3x^2 - 21x + 18 = 0$

$x^2 - 7x + 6 = 0$

$(x-6)(x-1) = 0$

$x = 6 \quad x = 1$

$3x^2 - 19x + 20 = -2x - 2$

$3x^2 - 17x + 22 = 0$

$(3x-11)(x-2) = 0$

$x = \frac{11}{3} \quad x = 2$

$1 < x < 2$

$\frac{11}{3} < x < 6$

3. $$y = \sqrt{8 + e^x}, \quad x \in \mathbb{R}$$

Find the series expansion for y in ascending powers of x, up to and including the term in x^2, giving each coefficient in its simplest form.

(8)

$f(x) = (8 + e^x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(8+e^x)^{-\frac{1}{2}} \times e^x$

$f''(x) = -\frac{1}{4}(8+e^x)^{-\frac{3}{2}} \times e^{2x} + \frac{1}{2}e^x(8+e^x)^{-\frac{1}{2}}$

$f(0) = (8+1)^{\frac{1}{2}} = 3$

$f'(0) = \frac{1}{2}(8+1)^{-\frac{1}{2}} \times 1 = \frac{1}{6}$

$f''(0) = -\frac{1}{4}(8+1)^{-\frac{3}{2}} + \frac{1}{2}(8+1)^{-\frac{1}{2}} = \frac{17}{108}$

$\therefore f(x) = 3 + \frac{1}{6}x + \frac{17}{216}x^2 \ldots$

blank

4. (a) Use de Moivre's theorem to show that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

(5)

(b) Hence solve for $0 \leqslant \theta \leqslant \frac{\pi}{2}$

$$64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 3 = 0$$

giving your answers as exact multiples of π.

(5)

$\cos 6\theta$ ∴

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$

$(\cos\theta + i\sin\theta)^6 = \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta$
$+ 6i\cos\theta\sin^5\theta - \sin^6\theta$

equating real parts
$\Rightarrow$

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-2\cos^2\theta+\cos^4\theta) - (1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$

$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$

b) $2\cos 6\theta - 1 = (64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 2) - 1$

$\therefore \cos 6\theta = \frac{1}{2} \Rightarrow 6\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\therefore \theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}$

5. (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 27e^{-x}$$

(6)

(b) Find the particular solution that satisfies $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$

(6)

CF $y = Ae^{mx}$ $\quad y'' + 2y' + 10y = 0$

$y' = Ame^{mx}$ $\quad Ae^{mx}(m^2 + 2m + 10) = 0$

$y'' = Am^2e^{mx}$ $\quad \neq 0 \quad = 0 \quad (m+1)^2 - 1 = -10$

$\Rightarrow m = -1 \pm 3i$

$\Rightarrow y_{cf} = e^{-x}[A\cos 3x + B\sin 3x]$

PI $y = \lambda e^{-x}$ $\quad 10y = 10\lambda e^{-x}$

$y' = -\lambda e^{-x}$ $\quad + 2y' = -2\lambda e^{-x}$

$y'' = \lambda e^{-x}$ $\quad + y'' = \lambda e^{-x}$

$27e^{-x} = 9\lambda e^{-x} \quad \therefore \lambda = 3 \quad y_{PI} = 3e^{-x}$

$\therefore y_{GS} = e^{-x}[A\cos 3x + B\sin 3x + 3]$

$x=0, y=0 \quad 0 = A + 3 \quad \therefore A = -3 \quad y_{GS} = e^{-x}[B\sin 3x - 3\cos 3x + 3]$

$y' = e^{-x}(3B\cos 3x + 3\sin 3x) + e^{-x}(B\sin 3x - 3\cos 3x + 3)$

$x=0, y'=0 \quad 0 = 3B - 3 + 3 \quad \therefore \underline{B = 0}$

$\therefore y = 3e^{-x}(1 - \cos 3x)$

6. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$, is given by

$$w = \frac{4(1-\mathrm{i})z - 8\mathrm{i}}{2(-1+\mathrm{i})z - \mathrm{i}}, \quad z \neq \frac{1}{4} - \frac{1}{4}\mathrm{i}$$

The transformation T maps the points on the line l with equation $y = x$ in the z-plane to a circle C in the w-plane.

(a) Show that

$$w = \frac{ax^2 + bx\mathrm{i} + c}{16x^2 + 1}$$

where a, b and c are real constants to be found.

(6)

(b) Hence show that the circle C has equation

$$(u-3)^2 + v^2 = k^2$$

where k is a constant to be found.

(4)

a) $z = x + iy \qquad x = y \Rightarrow z = x + ix = x(1+i)$

$$w = \frac{4(1-i)(1+i)x - 8i}{2(-1+i)(1+i)x - i} = \frac{(8x - 8i)}{(-4x - i)} \times \frac{(-4x+i)}{(-4x+i)}$$

$$\therefore w = \frac{-32x^2 + 40xi + 8}{16x^2 + 1}$$

z

b) $\therefore \quad w = u + iv \equiv \dfrac{8-32x^2}{16x^2+1} + \dfrac{40xi}{16x^2+1}$

$\therefore u = \dfrac{8-32x^2}{16x^2+1} \Rightarrow 16x^2u + u = 8 - 32x^2$

$16x^2u + 32x^2 = 8 - u$

$16x^2(u+2) = 8-u$

$16x^2 = \dfrac{8-u}{u+2}$

$\therefore 1600x^2 = 100\left(\dfrac{8-u}{u+2}\right)$

$16x^2+1 = \dfrac{8-\cancel{u}+\cancel{u}+2}{u+2}$

$16x^2+1 = \dfrac{10}{u+2}$

$v = \dfrac{40x}{16x^2+1} \Rightarrow v^2 = \dfrac{1600x^2}{(16x^2+1)^2} = 100\left(\dfrac{8-u}{u+2}\right)\left(\dfrac{u+2}{10}\right)^2$

$\Rightarrow v^2 = \dfrac{\cancel{100}(8-u)(u+2)^{\cancel{2}}}{\cancel{100}(\cancel{u+2})}$

$\therefore v^2 = (8-u)(u+2) \Rightarrow v^2 = -u^2 + 6u + 16$

$\Rightarrow u^2 - 6u + v^2 = 16$

$\Rightarrow (u-3)^2 + v^2 = 25 \qquad \therefore \underline{\underline{u = 5}}$

7. (a) Show that the substitution $v = y^{-3}$ transforms the differential equation

$$x\frac{dy}{dx} + y = 2x^4y^4 \qquad \text{(I)}$$

into the differential equation

$$\frac{dv}{dx} - \frac{3v}{x} = -6x^3 \qquad \text{(II)}$$

(5)

(b) By solving differential equation (II), find a general solution of differential equation (I) in the form $y^3 = f(x)$.

(6)

$V = y^{-3} \Rightarrow \frac{dV}{dx} = -3y^{-4}\frac{dy}{dx} \qquad \frac{dy}{dx} = -\frac{1}{3}y^4\frac{dV}{dx}$

$\Rightarrow x\frac{dy}{dx} + y = 2x^4y^4$

$= -\frac{1}{3}xy^4\frac{dV}{dx} + y = 2x^4y^4$

$\div\ xy^4 \quad -\frac{1}{3}\frac{dV}{dx} + \frac{1}{xy^3} = 2x^3 \quad (\times -3) \quad \frac{dV}{dx} - \frac{3}{xy^3} = -6x^3$

$\frac{1}{y^3} = V \quad \therefore \frac{dV}{dx} - \frac{3}{x}V = -6x^3$

$IF = e^{-\int\frac{3}{x}dx} = e^{-3\ln x} = (e^{\ln x})^{-3} = x^{-3} = \frac{1}{x^3}$

$\Rightarrow \frac{1}{x^3}\frac{dV}{dx} - \frac{3}{x^4}V = -6 \Rightarrow \frac{d}{dx}\left(\frac{V}{x^3}\right) = -6$

$\Rightarrow \frac{V}{x^3} = \int -6\,dx = -6x + c \quad \therefore V = cx^3 - 6x^4$

$V = \frac{1}{y^3} \therefore \frac{1}{y^3} = cx^3 - 6x^4 \quad \therefore y^3 = \frac{1}{cx^3 - 6x^4}$

8.

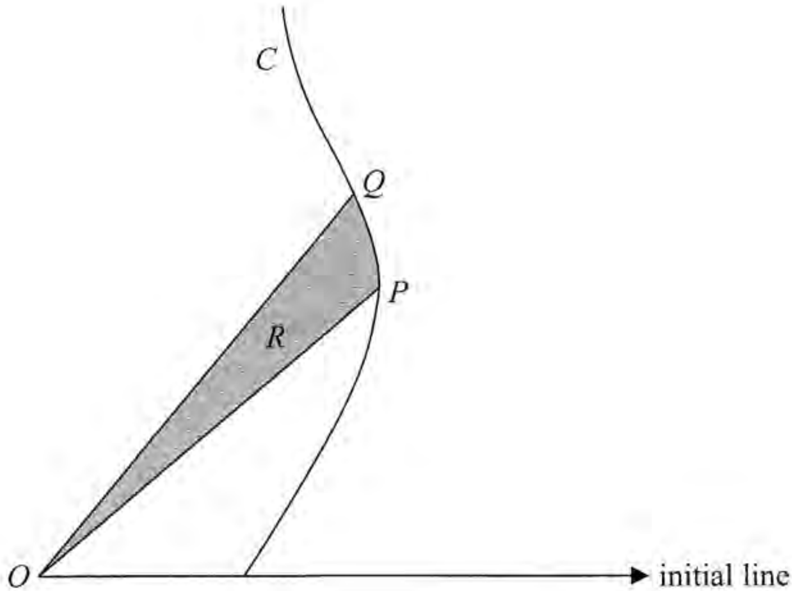


Figure 1

Figure 1 shows a sketch of part of the curve C with polar equation

$$r = 1 + \tan\theta, \quad 0 \leqslant \theta < \frac{\pi}{2}$$

The tangent to the curve C at the point P is perpendicular to the initial line.

(a) Find the polar coordinates of the point P.

(5)

The point Q lies on the curve C, where $\theta = \dfrac{\pi}{3}$

The shaded region R is bounded by OP, OQ and the curve C, as shown in Figure 1

(b) Find the exact area of R, giving your answer in the form

$$\frac{1}{2}(\ln p + \sqrt{q} + r)$$

where p, q and r are integers to be found.

(7)

$\frac{dy}{dx}=0 \qquad x = r\cos\theta = (1+\tan\theta)\cos\theta = \cos\theta + \frac{\sin\theta\cancel{\cos\theta}}{\cancel{\cos\theta}}$

$\therefore\ x = \cos\theta + \sin\theta$

$\frac{dx}{d\theta} = -\sin\theta + \cos\theta = 0 \Rightarrow \sin\theta = \cos\theta$

$\Rightarrow \tan\theta = 1 \ \therefore \theta = \frac{\pi}{4}$

$r = 1+\tan\frac{\pi}{4} \ \therefore r = 2 \qquad P\left(2, \frac{\pi}{4}\right)$

b) $\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1+\tan\theta)^2\,d\theta = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1+2\tan\theta+\tan^2\theta\,d\theta = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2\tan\theta + \sec^2\theta\,d\theta$

$= \frac{1}{2}\left[2\ln|\sec\theta| + \tan\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{2}\left[(2\ln 2+\sqrt{3}) - (2\ln\sqrt{2}+1)\right]$

$= \frac{1}{2}\left[2\ln\frac{2}{\sqrt{2}} + \sqrt{3} - 1\right] = \frac{1}{2}\left[\ln\left(\frac{2}{\sqrt{2}}\right)^2 + \sqrt{3} - 1\right]$

$= \frac{1}{2}\left[\ln 2 + \sqrt{3} - 1\right]$