FP2 S13 UK

1. (a) Express $\dfrac{2}{(2r+1)(2r+3)}$ in partial fractions.
(2)

(b) Using your answer to (a), find, in terms of n,

$$\sum_{r=1}^{n} \frac{3}{(2r+1)(2r+3)}$$

Give your answer as a single fraction in its simplest form.
(3)

$$\frac{2}{(2r+1)(2r+3)} = \frac{A}{2r+1} + \frac{B}{2r+3} \Rightarrow 2 = A(2r+3) + B(2r+1)$$

$$r = -\tfrac{1}{2} \Rightarrow A = 1 \qquad r = -\tfrac{3}{2} \quad B = -1$$

$$\therefore = \frac{1}{2r+1} - \frac{1}{2r+3}$$

b) $$\sum_{1}^{n} \frac{3}{(2r+1)(2r+3)} = \frac{3}{2}\sum_{1}^{n} \frac{2}{(2r+1)(2r+3)}$$

$$= \frac{3}{2}\left[\left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}\right) + \dots + \left(\cancel{\frac{1}{2n-1}} - \cancel{\frac{1}{2n+1}}\right) + \left(\cancel{\frac{1}{2n+1}} - \frac{1}{2n+3}\right)\right]$$

$$= \frac{3}{2}\left[\frac{1}{3} - \frac{1}{2n+3}\right] = \frac{\cancel{3}}{\cancel{2}}\left[\frac{\cancel{2}n + \cancel{3} - \cancel{3}}{\cancel{3}(2n+3)}\right] = \frac{n}{2n+3}$$

2.

$$z = 5\sqrt{3} - 5\text{i}$$

Find

(a) $|z|$,

(1)

(b) $\arg(z)$, in terms of π.

(2)

$$w = 2\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$$

Find

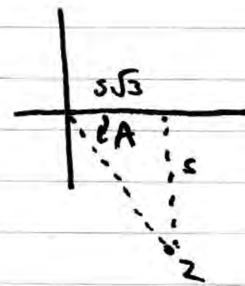
(c) $\left|\frac{w}{z}\right|$,

(1)

(d) $\arg\left(\frac{w}{z}\right)$, in terms of π.

(2)

a) $|z| = \sqrt{(5\sqrt{3})^2 + (5)^2} = 10$



b) $A = \tan^{-1}\left(\frac{5}{5\sqrt{3}}\right) = \frac{\pi}{6}$ $\therefore \arg(z) = -\frac{\pi}{6}$

$\left(\frac{11\pi}{6}\right)$

c) $z = 10\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)$

$$\frac{w}{z} = \frac{2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)}{10\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)} = \frac{1}{5}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)$$

$$\frac{\pi}{4} - \left(\frac{-\pi}{6}\right) = \frac{5\pi}{12}$$

$\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = \frac{2}{10} = \frac{1}{5}$ $\qquad \arg\left(\frac{w}{z}\right) = \frac{5\pi}{12}$

3.

$$\frac{d^2y}{dx^2} + 4y - \sin x = 0$$

Given that $y = \frac{1}{2}$ and $\frac{dy}{dx} = \frac{1}{8}$ at $x = 0$,

find a series expansion for y in terms of x, up to and including the term in x^3.

(5)

$x_0 = 0 \quad y_0 = \frac{1}{2} \quad y_0' = \frac{1}{8}$

$y'' + 4y - \sin x = 0 \Rightarrow y'' + 4\left(\frac{1}{2}\right) - \sin(0) \Rightarrow y'' = -2$

$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) + 4\frac{d}{dx}(y) - \frac{d}{dx}(\sin x) = 0 \Rightarrow y''' + 4y' - \cos x = 0$

$y''' + 4\left(\frac{1}{8}\right) - \cos(0) = 0 \Rightarrow y''' = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore y = \frac{1}{2} + \frac{1}{8}x - x^2 + \frac{1}{12}x^3$

blank

4. (a) Given that

$$z = r(\cos\theta + \mathrm{i}\sin\theta), \qquad r \in \mathbb{R}$$

prove, by induction, that $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta), \qquad n \in \mathbb{Z}^+$

(5)

$$w = 3\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)$$

(b) Find the exact value of w^5, giving your answer in the form $a + \mathrm{i}b$, where $a, b \in \mathbb{R}$.

(2)

$z^n = r^n(\cos n\theta + i\sin n\theta)$

$n=1 \quad z = r(\cos\theta + i\sin\theta) \qquad z^1 = r^1((\cos 1\theta + i\sin 1\theta)$

$z = r((\cos\theta + i\sin\theta)$ $\therefore$ true for $n=1$.

assume true for $n=k$ $\quad \therefore z^k = r^k(\cos k\theta + i\sin k\theta)$

$n=k+1 \qquad z^{k+1} = z^k \times z^1 = r^k((\cos k\theta + i\sin k\theta) \times r(\cos\theta + i\sin\theta)$

$= r^{k+1}\left[(\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\right]$

$= r^{k+1}\left[\cos(k\theta+\theta) + i\sin(k\theta+\theta)\right]$

$= r^{k+1}\left[\cos(k+1)\theta + i\sin(k+1)\theta\right] = z^{k+1}$ #

$\therefore$ true for $n=1$, true for $n=k+1$ if true for $n=k$

$\therefore$ by Mathematical induction true for all $n \in \mathbb{Z}$.

b) $w^5 = 3^5\left(\cos\frac{15\pi}{4} + i\sin\frac{15\pi}{4}\right) = 243\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$= 243\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = \frac{243\sqrt{2}}{2} + i\left(-\frac{243\sqrt{2}}{2}\right)$

5. (a) Find the general solution of the differential equation

$$x\frac{dy}{dx} + 2y = 4x^2$$

(5)

(b) Find the particular solution for which $y = 5$ at $x = 1$, giving your answer in the form $y = f(x)$.

(2)

(c) (i) Find the exact values of the coordinates of the turning points of the curve with equation $y = f(x)$, making your method clear.

(ii) Sketch the curve with equation $y = f(x)$, showing the coordinates of the turning points.

(5)

a) $\frac{dy}{dx} + \frac{2}{x}y = 4x$ IF $f(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = (e^{\ln x})^2$
$= x^2$

$\therefore x^2\frac{dy}{dx} + 2xy = 4x^3$

$\Rightarrow \frac{d}{dx}(x^2y) = 4x^3 \Rightarrow x^2y = \int 4x^3 dx \Rightarrow x^2y = x^4 + c$

$\therefore y = x^2 + cx^{-2}$

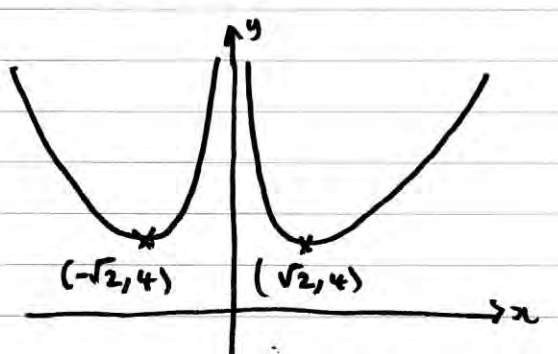
b) $(1,5)$ $5 = 1 + c$ $\therefore \underline{c=4}$

$y = x^2 + \frac{4}{x^2}$

c) $y' = 2x - \frac{8}{x^3}$ TP $y'=0$ $\frac{8}{x^3} = 2x \Rightarrow x^4 = 4 \Rightarrow x = \pm\sqrt{2}$

$y = 2 + \frac{4}{2} = 4$

$(\sqrt{2}, 4)$; $(-\sqrt{2}, 4)$



6. (a) Use algebra to find the exact solutions of the equation

$$|2x^2 + 6x - 5| = 5 - 2x$$

(6)

(b) On the same diagram, sketch the curve with equation $y = |2x^2 + 6x - 5|$ and the line with equation $y = 5 - 2x$, showing the x-coordinates of the points where the line crosses the curve.

(3)

(c) Find the set of values of x for which

$$|2x^2 + 6x - 5| > 5 - 2x$$

(3)

$2x^2+6x-5 = 5-2x$

$2x^2+8x-10=0$

$x^2+4x-5=0$

$(x+5)(x-1)=0$

$x=-5 \quad x=1$

$2x^2+6x-5 = 2x-5$

$2x^2+4x=0$

$2x(x+2)=0$

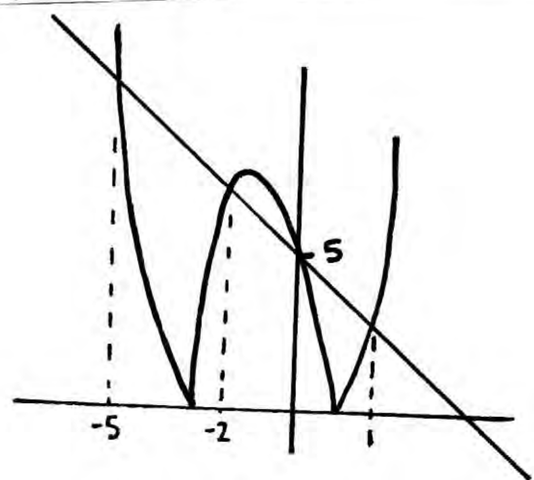
$x=0 \quad x=-2$

b) $2x^2+6x-5=0$

$x^2+3x-\frac{5}{2}=0$

$\left(x+\frac{3}{2}\right)^2=\frac{5}{2}+\frac{9}{4}=\frac{19}{4}$

$x=\frac{-3}{2}\pm\frac{\sqrt{19}}{2} \quad (-3.7, +0.7)$



c) $x<-5$ or $-2<x<0$ or $x>1$

7. (a) Show that the transformation $y = xv$ transforms the equation

$$4x^2\frac{d^2y}{dx^2} - 8x\frac{dy}{dx} + (8+4x^2)y = x^4 \qquad \text{(I)}$$

into the equation

$$4\frac{d^2v}{dx^2} + 4v = x \qquad \text{(II)}$$

(6)

(b) Solve the differential equation (II) to find v as a function of x.

(6)

(c) Hence state the general solution of the differential equation (I).

(1)

$y = vx \qquad \frac{dy}{dx} = x\frac{dv}{dx} + v$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(x\frac{dv}{dx}\right) + \frac{d}{dx}(v) = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$

$\Rightarrow 4x^2\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 8x\left(x\frac{dv}{dx} + v\right) + (8+4x^2)vx = x^4$

$= 4x^3\frac{d^2v}{dx^2} + \cancel{8x^2\frac{dv}{dx}} - \cancel{8x^2\frac{dv}{dx}} - \cancel{8xv} + \cancel{8xv} + 4x^3v = x^4$

$\div x^3 \qquad 4\frac{d^2v}{dx^2} + 4v = x$

b) $v = ax + b \qquad 4(0) + 4ax + 4b \equiv x \qquad \therefore b = 0 \quad a = \frac{1}{4}$

$v' = a \qquad v_{PI} = \frac{1}{4}x$

$v'' = 0$

$v = Ae^{mx} \qquad \Rightarrow \underbrace{Ae^{mx}}_{\neq 0}\underbrace{(4m^2 + 4)}_{=0} = 0 \quad \Rightarrow m^2 = -1 \quad m = \pm i$

$v' = Ame^{mx}$

$v'' = Am^2e^{mx} \qquad v = Pe^{ix} + Qe^{-ix} \Rightarrow v_{cf} = A\cos x + B\sin x$

$\therefore v_{GS} = A\cos x + B\sin x + \frac{1}{4}x$

c) $\frac{y}{x} = A\cos x + B\sin x + \frac{1}{4}x \Rightarrow y = Ax\cos x + Bx\sin x + \frac{1}{4}x^2$

8.

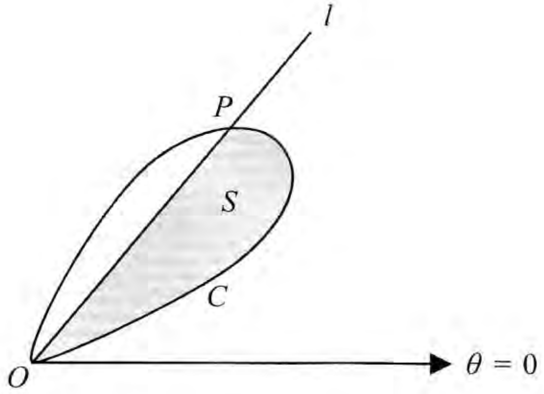


Figure 1

Figure 1 shows a curve C with polar equation $r = a\sin 2\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$, and a half-line l.

The half-line l meets C at the pole O and at the point P. The tangent to C at P is parallel to the initial line. The polar coordinates of P are (R, ϕ).

(a) Show that $\cos\phi = \frac{1}{\sqrt{3}}$

(6)

(b) Find the exact value of R.

(2)

The region S, shown shaded in Figure 1, is bounded by C and l.

(c) Use calculus to show that the exact area of S is

$$\frac{1}{36}a^2\left(9\arccos\left(\frac{1}{\sqrt{3}}\right) + \sqrt{2}\right)$$

(7)

parallel to initial line when $\frac{dy}{d\theta} = 0$ $\quad y = r\sin\theta = a\sin 2\theta\sin\theta$

$\frac{dy}{d\theta} = a\,2\cos 2\theta\sin\theta + a\sin 2\theta\cos\theta = 0$

$2(2\cos^2\theta - 1)\sin\theta = -2\cancel{\sin\theta}\cos^2\theta$

$4\cos^2\theta - 2 = -2\cos^2\theta \Rightarrow 6\cos^2\theta = 2 \therefore \cos\theta = \frac{1}{\sqrt{3}}$ #

b) $r = a\sin 2\theta = a\,2\sin\theta\cos\theta$

$= a\,2 \times \frac{\sqrt{2}}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = \frac{2\sqrt{2}}{3}a$

$\sqrt{3}$, $\sqrt{2}$, θ, 1 $\quad \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$

c) Area $= \frac{1}{2}a^2\int_0^{\arccos(\frac{1}{\sqrt{3}})}(\sin 2\theta)^2\,d\theta = \frac{1}{4}a^2\int_0^{\arccos(\frac{1}{\sqrt{3}})} 1 - \cos 4\theta\,d\theta = \frac{1}{4}a^2\left[\theta - \frac{1}{4}\sin 4\theta\right)_0^{\arccos(\frac{1}{\sqrt{3}})}$

$\cos 4\theta = 1 - 2\sin^2 2\theta$ $\quad = \frac{1}{16}a^2\left[4\theta - \sin 4\theta\right]_0^{\arccos(\frac{1}{\sqrt{3}})}$

$$= \frac{1}{16}a^2\left[4\theta - 2\sin 2\theta \cos 2\theta\right]_0^{\arccos\left(\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{1}{16}a^2\left[4\theta - 4\sin\theta\cos\theta\left(2\cos^2\theta - 1\right)\right]_0^{\arccos\left(\frac{1}{\sqrt{3}}\right)}$$

$$\frac{1}{16}a^2\left[4\left(\arccos\left(\frac{1}{\sqrt{3}}\right)\right) - 4\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{3} - 1\right)\right]$$

$$= \frac{1}{16}a^2\left[4\arccos\left(\frac{1}{\sqrt{3}}\right) + \frac{4\sqrt{2}}{9}\right]$$

$$= \frac{1}{9} \times \frac{4}{16}a^2\left[9\arccos\left(\frac{1}{\sqrt{3}}\right) + \sqrt{2}\right]$$

$$= \frac{1}{36}a^2\left[9\arccos\left(\frac{1}{\sqrt{3}}\right) + \sqrt{2}\right]$$