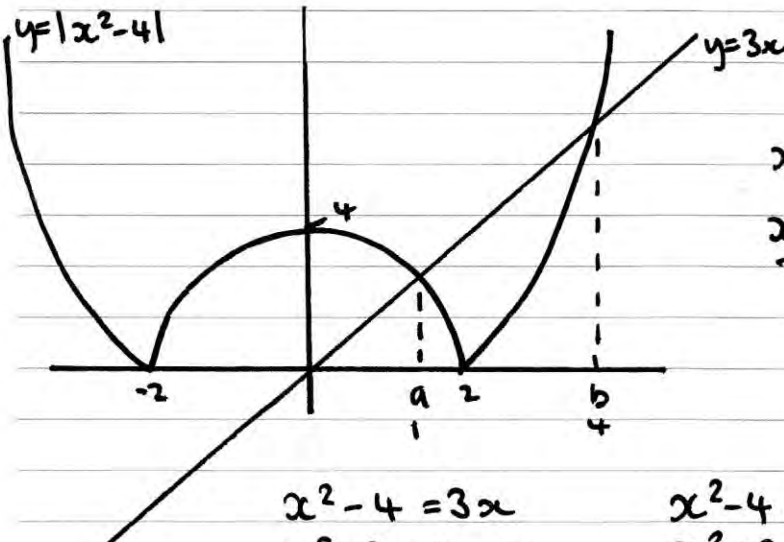
FP2 June 12

1. Find the set of values of x for which

$$|x^2 - 4| > 3x$$

(5)



$x < a$ or $x > b$

$\underline{x < 1}$ or $\underline{x > 4}$

$x^2 - 4 = 3x$

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$\underline{x = 4}$ $\cancel{x = -1}$

$x^2 - 4 = -3x$

$x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$\cancel{x = -4}$ $\underline{x = 1}$

2. The curve C has polar equation

$$r = 1 + 2\cos\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point P on C, the tangent to C is parallel to the initial line.

Given that O is the pole, find the exact length of the line OP.

(7)

parallel to Initial line $\Rightarrow \frac{dy}{d\theta} = 0$

$y = r\sin\theta = (1+2\cos\theta)\sin\theta$

$\frac{dy}{d\theta} = (1+2\cos\theta)\cos\theta - 2\sin\theta \times \sin\theta$

$\Rightarrow \cos\theta + 2\cos^2\theta - 2\sin^2\theta = 0$

$\Rightarrow 2\cos^2\theta + \cos\theta - 2 + 2\cos^2\theta = 0$

$\Rightarrow 4\cos^2\theta + \cos\theta - 2 = 0$

$\Rightarrow \cos\theta = \frac{-1 \pm \sqrt{1+32}}{8} = \frac{-1 \pm \sqrt{33}}{8} \quad \therefore \theta = 0.936^c$

$r = 1 + 2\cos(0.936\ldots) = 2.186\ldots$ (exact!)

$r = 1 + 2\left[\frac{-1+\sqrt{33}}{8}\right] = 1 + \frac{-1+\sqrt{33}}{4} = \frac{3+\sqrt{33}}{4} = OP$

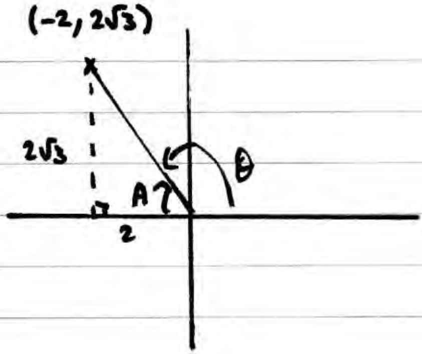
3. (a) Express the complex number $-2+\left(2\sqrt{3}\right)\mathrm{i}$ in the form $r(\cos\theta+\mathrm{i}\sin\theta),\ -\pi<\theta\leqslant\pi.$ **(3)**

(b) Solve the equation

$$z^4=-2+\left(2\sqrt{3}\right)\mathrm{i}$$

giving the roots in the form $r(\cos\theta+\mathrm{i}\sin\theta),\ -\pi<\theta\leqslant\pi.$

(5)



$A=\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$

$\therefore \theta=\frac{2\pi}{3}$ $\quad r=\sqrt{2^2+(2\sqrt{3})^2}=4$

$=4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$z^4=4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$\Rightarrow z=4^{\frac{1}{4}}\left(\cos\left(\frac{2\pi}{3}+2k\pi\right)+i\sin\left(\frac{2\pi}{3}+2k\pi\right)\right)^{\frac{1}{4}}$

$\Rightarrow z=\sqrt{2}\left[\cos\left(\left(\frac{6k+2}{3}\right)\pi\right)+i\sin\left(\frac{6k+2}{3}\right)\pi\right)^{\frac{1}{4}}$

$\Rightarrow z=\sqrt{2}\left[\cos\left(\frac{6k+2}{12}\right)\pi+i\sin\left(\frac{6k+2}{12}\right)\pi\right)$

$k=-2 \Rightarrow z=\sqrt{2}\left[\cos\frac{-10}{12}\pi+i\sin\frac{-10}{12}\pi\right]$

$k=-1 \Rightarrow z=\sqrt{2}\left[\cos\frac{-4}{12}\pi+i\sin\frac{4}{12}\pi\right]$

$k=0 \Rightarrow z=\sqrt{2}\left[\cos\frac{2\pi}{12}+i\sin\frac{2\pi}{12}\right]$

$k=1 \Rightarrow z=\sqrt{2}\left[\cos\frac{8\pi}{12}+i\sin\frac{8\pi}{12}\right]$

blank

4. Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2\cos t - \sin t$$

(9)

$x = a\cos t + b\sin t$

$x' = -a\sin t + b\cos t$

$x'' = -a\cos t + -b\sin t$

$x'' + 5x' + 6x \equiv 2\cos t - \sin t$

$-a\cos t - b\sin t$
$+5b\cos t - 5a\sin t$
$+6a\cos t + 6b\sin t$

$(5a+5b)\cos t + (5b-5a)\sin t = 2\cos t - \sin t$

$\Rightarrow \quad 5a+5b=2$

$5a+5b=1 \quad -$

$10b = 1$

$b = \frac{1}{10} \quad a = \frac{3}{10}$

$\frac{1}{2} + 5b = 2 \Rightarrow 5b = \frac{3}{2}$

$\therefore x_{PI} = \frac{3}{10}\cos t + \frac{1}{10}\sin t$

$x = Ae^{mt}$

$x' = Ame^{mt}$

$x'' = Am^2e^{mt}$

$x'' + 5x' + 6x = 0$

$\Rightarrow \underbrace{Ae^{mt}}_{\neq 0}\underbrace{(m^2+5m+6)}_{=0} = 0$

$\Rightarrow (m+3)(m+2) = 0$

$m=-3 \quad m=-2$

$\therefore x_{cf} = Ae^{-2t} + Be^{-3t}$

$\therefore x = \frac{3}{10}\cos t + \frac{1}{10}\sin t + Ae^{-2t} + Be^{-3t}$

5.

$$x\frac{dy}{dx} = 3x + y^2$$

(a) Show that

$$x\frac{d^2y}{dx^2} + (1-2y)\frac{dy}{dx} = 3$$

(2)

Given that $y = 1$ at $x = 1$,

(b) find a series solution for y in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$.

(8)

$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d}{dx}(3x) + \frac{d}{dx}(y^2)$

$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3 + 2y\frac{dy}{dx} \quad \Rightarrow x\frac{d^2y}{dx^2} + (1-2y)\frac{dy}{dx} = 3$

$x=1,\ y=1 \qquad xy' = 3x + y^2 \Rightarrow (1)y' = 3(1) + (1)^2$

$\therefore y' = 4$

$xy'' + (1-2y)y' = 3 \Rightarrow (1)y'' + (1-2(1))4 = 3$

$y'' = 3+4 \therefore y'' = 7$

$y = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2} + \frac{f'''(1)(x-1)^3}{6}$

$\therefore y = 1 + 4(x-1) + \frac{7}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$

$\frac{d}{dx}\left(x\frac{d^2y}{dx^2}\right) + \frac{d}{dx}\left[(1-2y)\frac{dy}{dx}\right] = \frac{d}{dx}(3)$

$x\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 2\frac{dy}{dx}\frac{dy}{dx} + (1-2y)\frac{d^2y}{dx^2} = 0$

$\Rightarrow xy''' - 2(y')^2 + (2-2y)y'' = 0 \quad (1)y''' - 2(4)^2 + (2-2(1))7 = 0$

$y''' = 32 \Rightarrow f'''(1) = 32$

$\therefore y = 1 + 4(x-1) + \frac{7}{2}(x-1)^2 + \frac{16}{3}(x-1)^3 \ldots$

Z

6. (a) Express $\dfrac{1}{r(r+2)}$ in partial fractions.

(2)

(b) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(an+b)}{4(n+1)(n+2)}$$

where a and b are constants to be found.

(6)

(c) Hence show that

$$\sum_{r=n+1}^{2n} \frac{1}{r(r+2)} = \frac{n(4n+5)}{4(n+1)(n+2)(2n+1)}$$

(3)

a) $\dfrac{1}{r(r+2)} = \dfrac{A}{r} + \dfrac{B}{r+2} \Rightarrow 1 = A(r+2) + B(r)$ $\quad r=0 \Rightarrow A = \frac{1}{2}$

$r=-2 \Rightarrow B = -\frac{1}{2}$

$\therefore \dfrac{1}{r(r+2)} = \dfrac{1}{2r} - \dfrac{1}{2(r+2)}$

$r=1$

b) $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)} = \left(\frac{1}{2} - \cancel{\frac{1}{6}}\right) + \left(\frac{1}{4} - \cancel{\frac{1}{8}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}\right) + \dots$

$\dots + \left(\cancel{\frac{1}{2(n-1)}} - \frac{1}{2(n+1)}\right) + \left(\cancel{\frac{1}{2n}} - \frac{1}{2(n+2)}\right)$

$\therefore \displaystyle\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} = \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$

$= \dfrac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)} = \dfrac{n(3n+5)}{4(n+1)(n+2)}$

c) $\displaystyle\sum_{r=n+1}^{2n} \frac{1}{r(r+2)} = \sum_{r=1}^{2n} \frac{1}{r(r+2)} - \sum_{r=1}^{n} \frac{1}{r(r+2)}$

$= \dfrac{\cancel{2}n(6n+5)}{4(2n+1)(\cancel{2}n+\cancel{2})} - \dfrac{n(3n+5)}{4(n+1)(n+2)} = \dfrac{n(6n+5)}{4(2n+1)(n+1)} - \dfrac{n(3n+5)}{4(n+1)(n+2)}$

$= \dfrac{n(6n+5)(n+2) - n(3n+5)(2n+1)}{4(n+1)(2n+1)(n+2)} = \dfrac{n[\cancel{6n^2} + 17n + 10 - \cancel{6n^2} - 13n - 5]}{4(n+1)(n+2)(2n+1)}$

$= \dfrac{n(4n+5)}{4(n+1)(n+2)(2n+1)}$

blank

7. (a) Show that the substitution $y = vx$ transforms the differential equation

$$3xy^2 \frac{dy}{dx} = x^3 + y^3 \qquad \text{(I)}$$

into the differential equation

$$3v^2 x \frac{dv}{dx} = 1 - 2v^3 \qquad \text{(II)}$$

(3)

(b) By solving differential equation (II), find a general solution of differential equation (I) in the form $y = f(x)$.

(6)

Given that $y = 2$ at $x = 1$,

(c) find the value of $\frac{dy}{dx}$ at $x = 1$

(2)

$y = vx \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx) = x\frac{dv}{dx} + v$

$\Rightarrow 3x(vx)^2\left(x\frac{dv}{dx} + v\right) = x^3 + (vx)^3$

$= 3x^4v^2\frac{dv}{dx} + 3x^3v^3 = x^3 + v^3x^3 \quad \div x^3$

$= 3xv^2\frac{dv}{dx} + 3v^3 = 1 + v^3 \quad \therefore 3v^2x\frac{dv}{dx} = 1 - 2v^3$

b) $\int \frac{3v^2}{1-2v^3}\,dv = \int \frac{1}{x}\,dx \Rightarrow \ln x + c = -\frac{1}{2}\int \frac{-6v^2}{1-2v^3}\,dv$

$\Rightarrow \ln x + c = -\frac{1}{2}\ln|1-2v^3| \Rightarrow \ln|1-2v^3| = -2\ln x - 2c$

$\Rightarrow \ln|1-2v^3| = \ln x^{-2} - 2c \Rightarrow 1-2v^3 = e^{\ln x^{-2} - 2c} = Ax^{-2} \quad A = e^{-2c}$

$\Rightarrow 2v^3 = 1 - Ax^{-2} \Rightarrow \frac{1}{2} + \frac{B}{x^2} = v^3 \qquad B = -\frac{A}{2}$

$C = -2B$

$\Rightarrow \frac{y^3}{x^3} = \frac{1}{2} + \frac{B}{x^2} \Rightarrow y^3 = \frac{1}{2}x^3 + Bx \quad \therefore y = \sqrt[3]{\frac{x^3 + Cx}{2}}$

c) $(1,2) \quad 2 = \sqrt[3]{\frac{1+C}{2}} \Rightarrow 8 = \frac{1+C}{2} \Rightarrow C = 15 \quad \therefore y = \sqrt[3]{\frac{x^3+15x}{2}}$ (not needed)

$3xy^2\frac{dy}{dx} = x^3 + y^3 \Rightarrow 3(1)(2)^2\frac{dy}{dx} = 1^3 + 2^3 \Rightarrow 12\frac{dy}{dx} = 9$

$\therefore \frac{dy}{dx} = \frac{3}{4}$

blank

8. The point P represents a complex number z on an Argand diagram such that

$$|z-6\mathrm{i}| = 2|z-3|$$

(a) Show that, as z varies, the locus of P is a circle, stating the radius and the coordinates of the centre of this circle.

(6)

The point Q represents a complex number z on an Argand diagram such that

$$\arg(z-6) = -\frac{3\pi}{4}$$

(b) Sketch, on the same Argand diagram, the locus of P and the locus of Q as z varies.

(4)

(c) Find the complex number for which both $|z-6\mathrm{i}| = 2|z-3|$ and $\arg(z-6) = -\frac{3\pi}{4}$

(4)

a) $|x+(y-6)i| = 2|(x-3)+yi|$

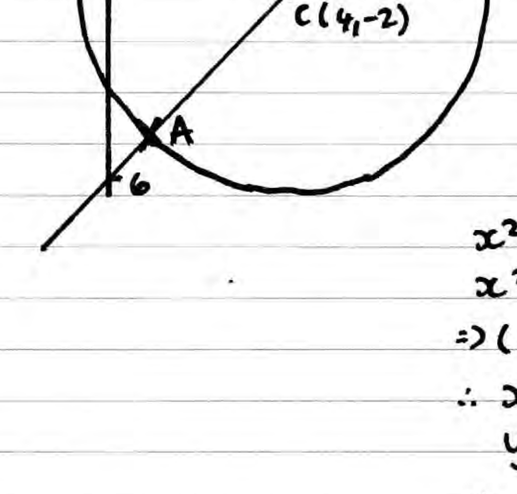
$\Rightarrow x^2+(y-6)^2 = 4[(x-3)^2+y^2] \Rightarrow x^2+y^2-12y+\cancel{36} = 4x^2-24x+\cancel{36}+4y^2$

$\Rightarrow 3x^2-24x+3y^2+12y=0 \Rightarrow x^2-8x+y^2+4y=0$

$\Rightarrow (x-4)^2+(y+2)^2=20$ Circle Centre $(4,-2)$ $r=\sqrt{20}$

≈ 4.5

b)



at $(0,0)$ $(-4^2)+(2)^2=20$

$\therefore$ passes through origin.

half line 45°

will pass through $(4,-2)$

half line $\Rightarrow y=x-6$ $x\le 6$

$(x-4)^2+((x-6)+2)^2=20$

$x^2-8x+16+x^2-8x+16=20$

$x^2-8x+16=10 \Rightarrow x^2-8x+6=0$

$\Rightarrow (x-4)^2=10 \Rightarrow x=4\pm\sqrt{10}$

$\therefore x=4-\sqrt{10}$ Since $x\le 6$

$y=x-6 \Rightarrow y=-2-\sqrt{10}$

$A \Rightarrow (4-\sqrt{10})+i(-2-\sqrt{10})$