

FP2 June 10

1. (a) Express $\frac{3}{(3r-1)(3r+2)}$ in partial fractions.

(2)

(b) Using your answer to part (a) and the method of differences, show that

$$\sum_{r=1}^n \frac{3}{(3r-1)(3r+2)} = \frac{3n}{2(3n+2)}$$

(3)

(c) Evaluate $\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)}$, giving your answer to 3 significant figures.

(2)

$$\frac{3}{(3r-1)(3r+2)} = \frac{A}{3r-1} + \frac{B}{3r+2} \Rightarrow 3 = A(3r+2) + B(3r-1) \quad r = \frac{1}{3} \quad A = 1$$
$$r = -\frac{2}{3} \quad B = -1$$

$$= \frac{1}{3r-1} + \frac{-1}{3r+2}$$

$$\text{b) } \sum_{r=1}^n \frac{3}{(3r-1)(3r+2)} = \left(\frac{1}{2} - \frac{1}{5}\right)_{r=1} + \left(\frac{1}{5} - \frac{1}{8}\right)_{r=2} + \left(\frac{1}{8} - \frac{1}{11}\right)_{r=3} + \dots + \left(\frac{1}{3n-4} - \frac{1}{3n-1}\right)_{r=n-1} + \frac{1}{3n-1} - \frac{1}{3n+2}_{r=n}$$
$$= \frac{1}{2} - \frac{1}{3n+2} = \frac{3n+2-2}{2(3n+2)} = \frac{3n}{2(3n+2)}$$

$$\text{c) } \sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)} = \frac{3000}{2(3002)} - \frac{3 \times 99}{2(3 \times 99 + 2)} = \frac{0.00301}{2}$$

2. The displacement x metres of a particle at time t seconds is given by the differential equation

$$\frac{d^2x}{dt^2} + x + \cos x = 0$$

When $t = 0$, $x = 0$ and $\frac{dx}{dt} = \frac{1}{2}$.

Find a Taylor series solution for x in ascending powers of t , up to and including the term in t^3 .

(5)

$$\frac{d}{dt} \left(\frac{d^2x}{dt^2} \right) + \frac{d}{dt}(x) + \frac{d}{dt}(\cos x) = 0$$

$$\Rightarrow \frac{d^3x}{dt^3} + \frac{dx}{dt} - \sin x \frac{dx}{dt} = 0 \Rightarrow \frac{d^3x}{dt^3} + (1 - \sin x) \frac{dx}{dt} = 0$$

$$t=0 \quad x_0=0 \quad x'_0 = \frac{1}{2}$$

$$x'' + x + \cos x = 0$$

$$x'' + 0 + 1 = 0 \Rightarrow x'' = -1$$

$$x''' + (1 - \sin x)x' = 0$$

$$x''' + (1 - 0)\frac{1}{2} = 0 \Rightarrow x''' = -\frac{1}{2}$$

$$\therefore x = \frac{1}{2}t - \frac{1}{2}t^2 - \frac{1}{12}t^3 \dots$$

3. (a) Find the set of values of x for which

$$x+4 > \frac{2}{x+3} \quad (6)$$

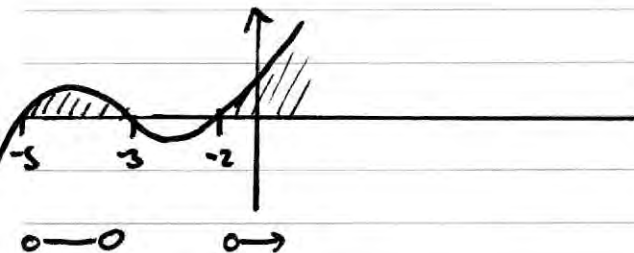
(b) Deduce, or otherwise find, the values of x for which

$$x+4 > \frac{2}{|x+3|} \quad (1)$$

$$(x+3)^2(x+4) > 2 \frac{(x+3)^2}{x+3} \Rightarrow (x+3)^2(x+4) - 2(x+3) > 0$$

$$\Rightarrow (x+3)[(x+3)(x+4) - 2] > 0 \Rightarrow (x+3)[x^2+7x+12-2] > 0$$

$$\Rightarrow (x+3)(x+5)(x+2) > 0$$



$$x > -2 \text{ or } -5 < x < -3$$

$$\text{b) } \underline{x > -2}$$

4.

$$z = -8 + (8\sqrt{3})i$$

(a) Find the modulus of z and the argument of z .

(3)

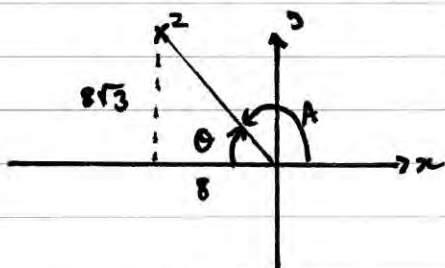
Using de Moivre's theorem,

(b) find z^3 ,

(2)

(c) find the values of w such that $w^4 = z$, giving your answers in the form $a + ib$, where $a, b \in \mathbb{R}$.

(5)



$$\tan \theta = \left(\frac{8\sqrt{3}}{8} \right) \Rightarrow \theta = \frac{\pi}{3} \Rightarrow A = \frac{2\pi}{3} = \arg(z)$$

$$|z| = \sqrt{8^2 + (8\sqrt{3})^2} = \underline{16}$$

$$b) z^3 = \left(16 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \right)^3 = 4096 \left(\cos(2\pi) + i \sin(2\pi) \right) = \underline{4096}$$

$$c) w = z^{\frac{1}{4}} = 16^{\frac{1}{4}} \left(\cos \left(\frac{2\pi}{3} + 2k\pi \right) + i \sin \left(\frac{2\pi}{3} + 2k\pi \right) \right)^{\frac{1}{4}}$$

$$w = 2 \left[\cos \left(\frac{6k+2}{3} \pi \right) + i \sin \left(\frac{6k+2}{3} \pi \right) \right]^{\frac{1}{4}}$$

$$w = 2 \left[\cos \left(\frac{3k+1}{6} \pi \right) + i \sin \left(\frac{3k+1}{6} \pi \right) \right] =$$

$$k=-2 \quad w = 2 \left[\cos \left(-\frac{5\pi}{6} \right) + i \sin \left(-\frac{5\pi}{6} \right) \right] = -\sqrt{3} - i$$

$$k=-1 \quad w = 2 \left[\cos \left(-\frac{2\pi}{6} \right) + i \sin \left(-\frac{2\pi}{6} \right) \right] = 1 - \sqrt{3}i$$

$$k=0 \quad w = 2 \left[\cos \left(\frac{\pi}{6} \right) + i \sin \left(\frac{\pi}{6} \right) \right] = \sqrt{3} + i$$

$$k=1 \quad w = 2 \left[\cos \left(\frac{4\pi}{6} \right) + i \sin \left(\frac{4\pi}{6} \right) \right] = -1 + \sqrt{3}i$$

5.

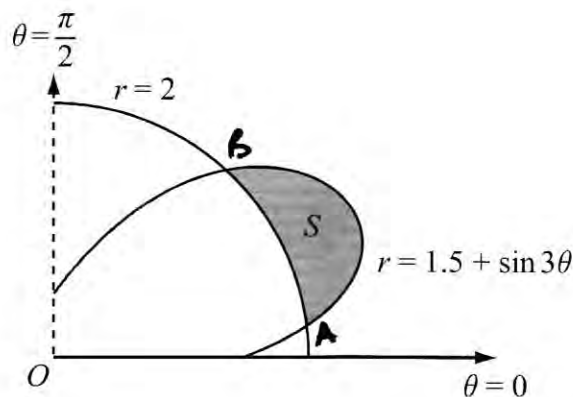


Figure 1

Figure 1 shows the curves given by the polar equations

$$r = 2, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

$$\text{and } r = 1.5 + \sin 3\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

(a) Find the coordinates of the points where the curves intersect.

(3)

The region S , between the curves, for which $r > 2$ and for which $r < (1.5 + \sin 3\theta)$, is shown shaded in Figure 1.

(b) Find, by integration, the area of the shaded region S , giving your answer in the form $a\pi + b\sqrt{3}$, where a and b are simplified fractions.

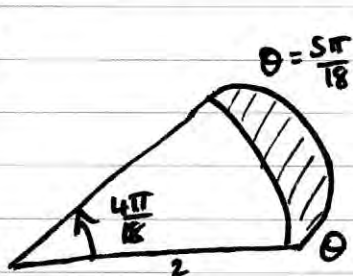
(7)

$$a) \quad 2 = 1.5 + \sin 3\theta \Rightarrow \sin 3\theta = \frac{1}{2} \Rightarrow 3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{18}, \frac{5\pi}{18}$$

$$A(2, \frac{\pi}{18}); B(2, \frac{5\pi}{18})$$

b)



$$S = \frac{1}{2} \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} (1.5 + \sin 3\theta)^2 d\theta - \text{Sector}$$

$$S = \frac{1}{2} \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \frac{9}{4} + 3\sin 3\theta + \sin^2 3\theta d\theta - \frac{1}{2}(2)^2 \left(\frac{4\pi}{9}\right)$$

$$\cos 6\theta = 1 - 2\sin^2 3\theta$$

$$S = \frac{1}{2} \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \frac{9}{4} + 3\sin 3\theta + \left(\frac{1}{2} - \frac{1}{2}(\cos 6\theta)\right) d\theta - \frac{4\pi}{9}$$

$$= \frac{1}{2} \left[\frac{11}{4}\theta - \cos 3\theta - \frac{1}{12}\sin 6\theta \right]_{\frac{\pi}{18}}^{\frac{5\pi}{18}} - \frac{4\pi}{9} = \frac{1}{2} \left[\left(\frac{55\pi}{72} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{24}\right) - \left(\frac{11\pi}{72} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{24}\right) \right] - \frac{4\pi}{9}$$

$$= \frac{13\sqrt{3}}{24} - \frac{5\pi}{36}$$

6. A complex number z is represented by the point P in the Argand diagram.

(a) Given that $|z-6|=|z|$, sketch the locus of P .

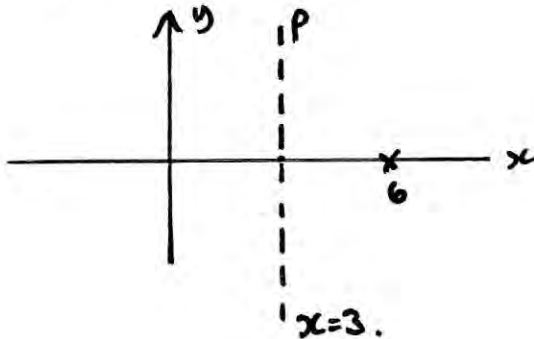
(2)

(b) Find the complex numbers z which satisfy both $|z-6|=|z|$ and $|z-3-4i|=5$.

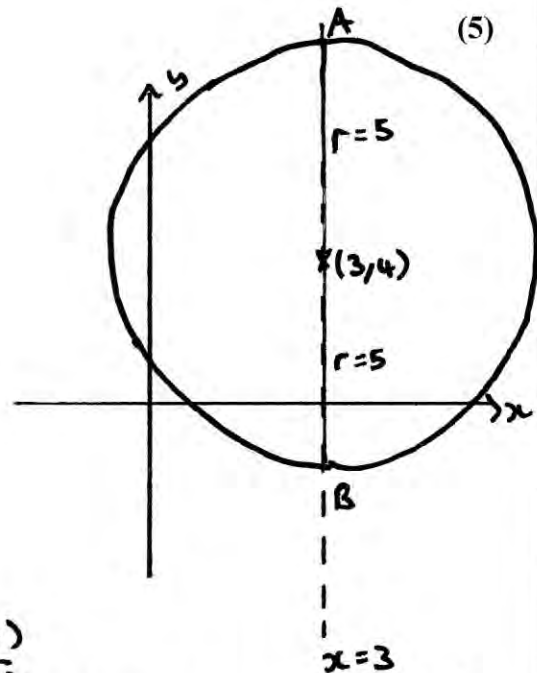
(3)

The transformation T from the z -plane to the w -plane is given by $w = \frac{30}{z}$.

(c) Show that T maps $|z-6|=|z|$ onto a circle in the w -plane and give the cartesian equation of this circle.



b)



$$c) z = \frac{30}{w} \Rightarrow 3 = \frac{30}{u+iv}$$

$$3 = \frac{30(u-iv)}{(u+iv)(u-iv)} = \frac{30u}{u^2+v^2} + i \frac{(-30v)}{u^2+v^2}$$

$$\text{real part} = 3 \Rightarrow 3 = \frac{30u}{u^2+v^2}$$

$$\Rightarrow 3u^2 + 3v^2 = 30u \Rightarrow 3u^2 - 30u + 3v^2 = 0 \Rightarrow u^2 - 10u + v^2 = 0$$

$$\Rightarrow (u-5)^2 + v^2 = 25 \quad x=3 \Rightarrow \text{maps to a circle centre } (5,0) \text{ } r=5 \text{ in the } w\text{-plane.}$$

$$(A) 3+9i \quad (B) 3-i$$

alt

$$|z-6|=|z| \Rightarrow \left| \frac{30}{w} - 6 \right| = \left| \frac{30}{w} \right| \Rightarrow \left| \frac{30-6w}{w} \right| = \left| \frac{30}{w} \right|$$

$$\Rightarrow |30-6w| = |30| \Rightarrow 6|5-w| = 6|5| \Rightarrow |5-w| = |5|$$

$$\Rightarrow |w-5| = |5| \Rightarrow |u+iv-5| = |5| = |(u-5)+iv| = |5|$$

$$\Rightarrow (u-5)^2 + v^2 = 25$$

7. (a) Show that the transformation $z = y^{\frac{1}{2}}$ transforms the differential equation

$$\frac{dy}{dx} - 4y \tan x = 2y^{\frac{1}{2}} \quad (I)$$

into the differential equation

$$\frac{dz}{dx} - 2z \tan x = 1 \quad (II) \quad (5)$$

(b) Solve the differential equation (II) to find z as a function of x . (6)

(c) Hence obtain the general solution of the differential equation (I). (1)

$$z = y^{\frac{1}{2}} \Rightarrow y = z^2 \Rightarrow \frac{dy}{dx} = 2z \frac{dz}{dx}$$

$$\Rightarrow 2z \frac{dz}{dx} - 4z^2 \tan x = 2z \quad (\div 2z)$$

$$\Rightarrow \frac{dz}{dx} - 2z \tan x = 1$$

$$\begin{aligned} \text{b) } \frac{dz}{dx} - (2 \tan x)z &= 1 & \text{IF } f(x) &= e^{\int -2 \tan x} = e^{-2 \ln |\sec x|} \\ & & &= (e^{\ln |\sec x|})^{-2} = (\sec x)^{-2} \\ \cos^2 x \frac{dz}{dx} - \cos^2 x (2 \tan x)z &= \cos^2 x & &= \underline{\cos^2 x} \end{aligned}$$

$$\Rightarrow \frac{d}{dx} (z \cos^2 x) = \cos^2 x \Rightarrow z \cos^2 x = \int \cos^2 x dx$$

$$\Rightarrow z \cos^2 x = \frac{1}{2} \int \cos 2x + 1 = \frac{1}{4} \sin 2x + \frac{1}{2} x + C$$

$$z = \frac{\sin 2x}{4 \cos^2 x} + \frac{x}{2 \cos^2 x} + \frac{C}{\cos^2 x}$$

$$\text{c) } z = y^{\frac{1}{2}} \quad \therefore y = \left(\frac{\sin 2x}{4 \cos^2 x} + \frac{x}{2 \cos^2 x} + \frac{C}{\cos^2 x} \right)^2$$

$$\sin 2x = 2 \sin x \cos x$$

$$\Rightarrow y = \left(\frac{1}{2} \tan x + \frac{1}{2} x \sec^2 x + C \sec^2 x \right)^2$$

8. (a) Find the value of λ for which $y = \lambda x \sin 5x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 25y = 3 \cos 5x \tag{4}$$

(b) Using your answer to part (a), find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 25y = 3 \cos 5x \tag{3}$$

Given that at $x = 0$, $y = 0$ and $\frac{dy}{dx} = 5$,

(c) find the particular solution of this differential equation, giving your solution in the form $y = f(x)$. (5)

(d) Sketch the curve with equation $y = f(x)$ for $0 \leq x \leq \pi$. (2)

$$y = \lambda x \sin 5x$$

$$y' = 5\lambda x \cos 5x + \lambda \sin 5x$$

$$y'' = -25\lambda x \sin 5x + 5\lambda \cos 5x + 5\lambda \cos 5x = -25\lambda x \sin 5x + 10\lambda \cos 5x$$

$$y'' + 25y = \cancel{-25\lambda x \sin 5x} + 10\lambda \cos 5x + 25\lambda x \sin 5x = 3 \cos 5x$$

$$\therefore 10\lambda = 3 \quad \lambda = \frac{3}{10}$$

$$y_{PI} = \frac{3}{10} x \sin 5x$$

$$y = Ae^{mt} \quad y'' + 25y = 0$$

$$y' = Ame^{mt}$$

$$y'' = Am^2e^{mt}$$

$$Ae^{mt}(m^2 + 25) = 0$$

$$\neq 0 \quad = 0 \quad \Rightarrow m = \pm 5i$$

$$y_{CF} = A(\cos 5x + B \sin 5x) \quad \therefore y = A \cos 5x + (B + \frac{3}{10}x) \sin 5x$$

$$x=0, y=0 \Rightarrow 0 = A$$

$$y = (B + \frac{3}{10}x) \sin 5x$$

$$x=0, y'=5 \Rightarrow 5 = 5B \Rightarrow \underline{B=1}$$

$$y' = 5(B + \frac{3}{10}x) \cos 5x + \frac{3}{10} \sin 5x$$

$$\therefore y = \underline{\underline{(1 + \frac{3}{10}x) \sin 5x}}$$

$$y = \left(1 + \frac{3}{10}x\right) \sin 5x$$

