# REVISION SHEET – FP1 (EDEXCEL)

# COMPLEX NUMBERS

**The main ideas are:**

- Manipulating complex numbers
- Complex conjugates and roots of equations
- The Argand diagram

***Before the exam you should know:***

- How to manipulate complex numbers and be able to multiply two complex numbers quickly and in one step as this will save a lot of time in the exam.
- How to geometrically interpret $|z_1 - z_2|$ as the distance between the complex numbers $z_1$ and $z_2$ in the Argand diagram.
- The fact that $|z_1 + z_2| = |z_1 - (-z_2)|$ which equals the distance between $z_1$ and $-z_2$ in the Argand diagram.

## Manipulating Complex Numbers.

### Multiplying, dividing, adding and subtracting

- Multiplying, adding and subtracting are all fairly straightforward.
- Dividing is slightly more complicated. Whenever you see a complex number on the denominator of a fraction you can "get rid of it" by multiplying both top and bottom of the fraction by its complex conjugate.

  e.g. $\dfrac{3+2i}{1-i} = \left(\dfrac{3+2i}{1-i}\right)\left(\dfrac{1+i}{1+i}\right) = \dfrac{1+5i}{2}$

## Complex Conjugates and Roots of Equations

### The complex conjugate

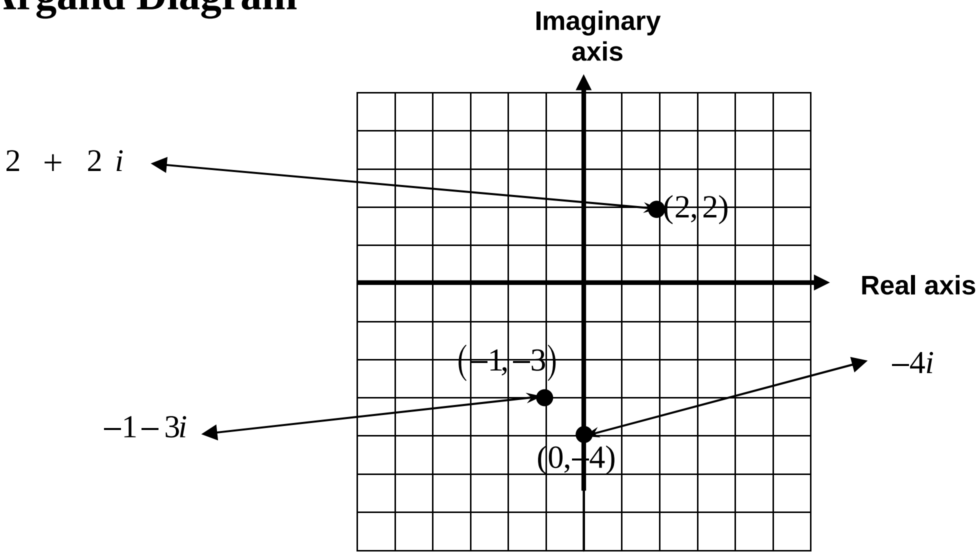
The complex conjugate of $z = a + bi$ is $z* = a - bi$.

- Remember $zz*$ is a real number and it equals the square of the modulus of z.
- Complex roots of polynomial equations with real coefficients occur in conjugate pairs. This means that if you are told one complex root of a polynomial equation with real coefficients you are in fact being told two roots. This is key to answering some typical exam questions.

An example of an algebraic trick that it is very useful to know is:

$$\begin{aligned}(z-(3+2i))(z-(3-2i)) &= ((z-3)-2i)((z-3)+2i) \\ &= (z-3)^2 - 4i^2 \\ &= z^2 - 6z + 13\end{aligned}$$

# The Argand Diagram




- In the Argand diagram the point (*x*, *y*) corresponds to the complex number *x* + *yi*.
- The argument of a complex number z, denoted $\arg(z)$ is the angle it makes with the positive real axis in the Argand diagram, measured anticlockwise and such that $-\pi < \arg(z) \leq \pi$ .
- When answering exam questions about points in the Argand diagram be prepared to used geometrical arguments based around equilateral triangles, similar triangles, isosceles triangles and parallel lines to calculate lengths and angles. This particularly the case when referring to sums, products and quotients of complex numbers.

# Modulus-Argument (or Polar) Form

If $z = x + yi$ has $|z| = r$ and $\arg(z) = \theta$ then $z = r(\cos\theta + i\sin\theta)$ . This is called the *polar* or *modulus-argument* form.

**Example** Write $z = 3 - 3\mathrm{j}$ in polar form.

**Solution**

$|z| = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$ and $\arg(z) = \frac{3\pi}{4}$ .

Therefore in polar form z is $z = 3\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

**Example**

If $z = 6\left(\cos\left(\frac{3\pi}{7}\right) + i\sin\left(\frac{3\pi}{7}\right)\right)$, what are $|z|$ and $\arg(z)$ ?

**Solution**

Since z is given in polar form it can just be read off that $|z| = 6$ and $\arg(z) = \frac{3\pi}{7}$ .

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# NUMERICAL METHODS

## The main ideas are:

- **Using the method of interval bisection, linear interpolation and the Newton Raphson Method to approximate the solution of an equation.**
- **Knowing how accurate each of these methods is so that you are able to estimate error.**

***Before the exam you should know:***

- The formula associated with linear interpolation
$$c = \frac{a\mathrm{f}(b) - b\mathrm{f}(a)}{\mathrm{f}(b) - \mathrm{f}(a)}$$
- The formula associated with the Newton Raphson Method: $x_{r+1} = x_r - \dfrac{\mathrm{f}(x_r)}{\mathrm{f}'(x_r)}$
- For every one of the methods, you should know how to judge when you have the approximation to the required degree of accuracy and then check that this is indeed the case.

## The Bisection Method

If a root of an equation f($x$) = 0 lies between $a$ and $b$, then $c = \dfrac{a+b}{2}$ gives an approximation to the root. You can then determine whether the root lies between $a$ and $c$ or between $b$ and $c$ and repeat this idea, obtaining better and better interval estimates to the root. Eventually you can give an approximation of the root to the desired degree of accuracy.

**Example**

Show that the equation $x^3 = 5 - 3x$ has a root α between 1 and 2. Starting with these two points straddling α, apply the method of bisection 3 times and state what you find having done that.

**Solution**

First of all rearrange the equation into the form $x^3 + 3x - 5 = 0$. Any root of this equation is a root of the one we started with and vice versa.

Let $\mathrm{f}(x) = x^3 + 3x - 5$. Since f(1) = −1 and f(2) = 9, there must be a root of f($x$) = 0 between 1 and 2. The table below shows 3 applications of the method of bisection beginning with $a_1 = 1, b_1 = 2$.

| $r$ | $a_r$ | Sign of f($a_r$) | $b_r$ | Sign of f($b_r$) | $c_r$ =($a_r$+$b_r$)/2 | Sign of f( $c_r$) |
|---|---|---|---|---|---|---|
| 1 | 1 | -ve | 2 | +ve | 1.5 | +ve |
| 2 | 1 | -ve | 1.5 | +ve | 1.25 | +ve |
| 3 | 1 | -ve | 1.25 | +ve | 1.125 | -ve |

At this point you see that the root lies between 1.125 and 1.25. The function is negative when $x$ = 1.125 and positive when $x$ = 1.25.

## Linear Interpolation

If a root of an equation f($x$) = 0 lies between $a$ and $b$, then $c=\dfrac{a\mathrm{f}(b)-b\mathrm{f}(a)}{\mathrm{f}(b)-\mathrm{f}(a)}$ gives an approximation of the root. If needed you can determine whether the root lies between $a$ and $c$ or between $b$ and $c$ and repeat this process to obtain a sequence of approximations to the root.

**Example**

Show that the equation $\mathrm{f}(x)=x^2-7=0$ has a root, α, between 2 and 3. By using linear interpolation, starting with these two values, find an estimate to this root.

**Solution**

Since $\mathrm{f}(2)=-5<0$ and $\mathrm{f}(3)=2>0$ the function must cross the $x$-axis between $x$ = 2 and $x$ = 3 and so there will be a root of f($x$) = 0 there. With $a=2$ and $b=3$, the linear interpolation gives a first approximation to the solution of $c=\dfrac{a\mathrm{f}(b)-b\mathrm{f}(a)}{\mathrm{f}(b)-\mathrm{f}(a)}$.

In this case this is $\dfrac{(2\times\mathrm{f}(3))-(3\times\mathrm{f}(2))}{\mathrm{f}(3)-\mathrm{f}(2)}=\dfrac{(2\times2)-(3\times-3)}{2-(-3)}=\dfrac{13}{5}=2.6$.

Note: Since $\mathrm{f}(2.6)=-0.24$ the points 2.6 and 3 straddle the solution. Therefore $2.6\le\alpha\le3$ and you can apply the procedure again by calculating $x_2=\dfrac{a\mathrm{f}(b)-b\mathrm{f}(a)}{\mathrm{f}(b)-\mathrm{f}(a)}$ with $a=2.6$ and $b=3$.

## Newton Raphson Method

The sequence of values generated by $x_{r+1}=x_r-\dfrac{\mathrm{f}(x_r)}{\mathrm{f}'(x_r)}$ with $x_0$ an appropriate estimate, usually converges to a root of $\mathrm{f}(x)=0$ near to $x_0$.

**Example**

Use the Newton Raphson method to find the root the equation $x^4+x-3=0$ near $x$ = 1.5. With $x_0=1.5$, use the method three times (in other words calculate as far as $x_4$). Hence, give an approximation to the root and state its accuracy.

**Solution**

The iterative formula in the case of $\mathrm{f}(x)=x^4+x-3$ is as follows: $x_{r+1}=x_r-\dfrac{\mathrm{f}(x_r)}{\mathrm{f}'(x_r)}=x_r-\left(\dfrac{x_r{}^4+x_r-3}{4x_r{}^3+1}\right)$.

Thus $x_1=x_0-\left(\dfrac{x_0{}^4+x_0-3}{4x_0{}^3+1}\right)=1.5-\left(\dfrac{1.5^4+1.5-3}{(4\times1.5^3)+1}\right)=1.254310$

Similarly $x_2=x_1-\left(\dfrac{x_1{}^4+x_1-3}{4x_1{}^3+1}\right)=1.172278$, $x_3=1.164110$, and $x_4=1.164035$.

From this evidence it looks as though 1.164 may be an estimate to the root which is correct to three decimal places. You can check this by verifying that the function changes sign between 1.1635 and 1.1645. In fact

$$\mathrm{f}(1.1635)=-0.00391<0 \text{ and } \mathrm{f}(1.1645)=0.003399>0.$$

REVISION SHEET – FP1 (Edx)

# CO-ORDINATE SYSTEMS

**The main ideas are:**

- Parametric and Cartesian Equations of the parabola and rectangular hyperbola
- Equations of tangents and normals to the above
- Intrinsic coordinates and radius of curvature

***Before the exam you should know:***

- The parabola, ellipse and hyperbola are each loci of a point $P$ which moves so that its distance from the fixed point (the focus) is in a constant ration ($e$, the eccentricity) to its distance from a fixed line (the directrix).

## The Parabola

The parabola with equation $y^2 = 4ax$ has focus at $(a, 0)$ and directrix $x = -a$. Parametrically the parabola with equation $y^2 = 4ax$ is given by $x = at^2$, $y = 2at$.

The tangent at $(h, k)$ to the parabola has the equation $ky = 2a(x + h)$ and the tangent at $(at^2, 2at)$ has the equation $ty = x + at^2$. The corresponding normal has the equation $y + tx = 2at + at^3$.

**Example**

Find the equation of the tangent to the parabola with equation $y^2 = 4ax$ at the point $T(at^2, 2at)$. If $S$ is the focus find the equation of the chord $QSR$ which is parallel to the tangent at $T$. Prove that $QR = 4TS$.

**Solution**

The gradient of the tangent at $T$ is $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2a \times \frac{1}{2at} = \frac{1}{t}$.

The tangent passes through $T(at^2, 2at)$ and therefore has equation $y - 2at = \frac{1}{t}\left(x - at^2\right)$ or $ty = x + at^2$.

The chord QSR is parallel to this tangent and so has the same gradient. Since the chord passes through the focus $(a, 0)$ the equation of chord QSR is $y - 0 = \frac{1}{t}(x - a)$ or $ty = x - a$.

The distance from $T(at^2, 2at)$ to $S(a, 0)$ is

$$\begin{aligned}\sqrt{\left(at^2 - a\right)^2 + \left(2at - 0\right)^2} &= \sqrt{a^2t^4 - 2a^2t^2 + a^2 + 4a^2t^2} \\ &= \sqrt{a^2t^4 + 2a^2t^2 + a^2} \\ &= a\sqrt{t^4 + 2t^2 + 1} \\ &= a\left(t^2 + 1\right)\end{aligned}$$

Q and R are where $ty = x - a$ intersects $y^2 = 4ax$. Using $x = a + ty$ in $y^2 = 4ax$ gives $y^2 = 4a(a + ty)$ or

$y^2 - 4aty - 4a^2 = 0$. The formula for the roots of a quadratic gives $\frac{4at \pm \sqrt{16a^2t^2 + 16a^2}}{2} = 2at \pm 2a\sqrt{t^2 + 1}$.

The corresponding $x$–coordinates are $a+2at^2 \pm 2at\sqrt{t^2+1}$. The distance between the two points $\left(a+2at^2+2at\sqrt{t^2+1}, 2at+2a\sqrt{t^2+1}\right)$ and $\left(a+2at^2-2at\sqrt{t^2+1}, 2at-2a\sqrt{t^2+1}\right)$ is

$$\begin{aligned}\sqrt{16a^2t^2\left(t^2+1\right)+16a^2\left(t^2+1\right)} &= 4a\sqrt{\left(t^2+1\right)\left(t^2+1\right)} \\ &= 4a\left(t^2+1\right)\end{aligned}$$

So the distance from $T$ to $S$ is four times the distance from $Q$ to $R$.

# The Rectangular Hyperbola

The rectangular hyperbola is a special case of a hyperbola with $e = \sqrt{2}$. You should know the Cartesian and parametric equations for a hyperbola and the equations of tangents and normals.

# Intrinsic Coordinates

If the length of the arc $AP$ on a curve is s, and the tangent to the curve at $P$ makes an angle of $\psi$ with the positive x-axis, then $(s, \psi)$ are called the intrinsic coordinates of the point $P$. In particular

$$\frac{dy}{dx} = \tan\psi, \frac{dx}{ds} = \cos\psi, \frac{dy}{ds} = \sin\psi$$

The radius of curvature $\rho$ at point $P(x, y)$ on the curve is $\dfrac{ds}{d\psi}$ or $\dfrac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$ or $\dfrac{\left[\dot{x}^2+\dot{y}^2\right]^{\frac{3}{2}}}{\dot{x}\ddot{y}-\ddot{x}\dot{y}}$.

# REVISION SHEET – FP1 (EDEXCEL)

# SERIES

**The main ideas are:**

- Summing Series using standard formulae
- Telescoping

***Before the exam you should know:***

- The standard formula:

$$\sum_{r=1}^{n} r, \sum_{r=1}^{n} r^2, \sum_{r=1}^{n} r^3$$

- And be able to spot that a series like

$$(1\times 2)+(2\times 3)+\ldots+n(n+1)$$

can be written in sigma notation as:

$$\sum_{r=1}^{n} r(r+1)$$

## Summing Series

### Using standard formulae

Fluency is required in manipulating and simplify standard formulae sums like:

$$\sum_{r=1}^{n} r\left(r^2+1\right)=\sum_{r=1}^{n} r^3+\sum_{r=1}^{n} r=\frac{n^2\left(n+1\right)^2}{4}+\frac{n\left(n+1\right)}{2}$$

$$=\frac{1}{4}n\left(n+1\right)\left[n\left(n+1\right)+2\right]$$

$$=\frac{1}{4}n\left(n+1\right)\left(n^2+n+2\right).$$