FP11 May 2011

1. $$f(x) = 2x^3 - 6x^2 - 7x - 4$$

(a) Show that $f(4) = 0$ **(1)**

(b) Use algebra to solve $f(x) = 0$ completely. **(4)**

a) $f(4) = 2(4)^3 - 6(4)^2 - 7(4) - 4$

$= 128 - 96 - 28 - 4$

$= 0$ #

$\therefore (x-4)$ is a factor of $f(x)$.

b)

	$2x^2$	$+2x$	$+1$
x	$2x^3$	$+2x^2$	$+x$
-4	$-8x^2$	$-8x$	-4

$r=0$ ✓

$2x^2+2x+1=0$

$\div 2$ $\quad x^2+x=-\frac{1}{2}$

$\left(x+\frac{1}{2}\right)^2-\frac{1}{4}=-\frac{1}{2}$

$\left(x+\frac{1}{2}\right)^2=-\frac{1}{4}$

$\Rightarrow x+\frac{1}{2}=\pm\sqrt{\frac{-1}{4}}$

$\Rightarrow x+\frac{1}{2}=\pm\frac{1}{2}i \quad \Rightarrow x=-\frac{1}{2}\pm\frac{1}{2}i$

Solutions are $4,\ -\frac{1}{2}+\frac{1}{2}i,\ -\frac{1}{2}-\frac{1}{2}i$

2. (a) Given that

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 3 \\ 4 & 5 & 5 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix}$$

find **AB**.

(2)

(b) Given that

$$\mathbf{C} = \begin{pmatrix} 3 & 2 \\ 8 & 6 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 5 & 2k \\ 4 & k \end{pmatrix}, \text{ where } k \text{ is a constant}$$

and

$$\mathbf{E} = \mathbf{C} + \mathbf{D}$$

find the value of k for which **E** has no inverse.

(4)

a) $AB = \begin{pmatrix} 3 & 1 & 3 \\ 4 & 5 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 9 & 9 \end{pmatrix}$

b) $C + D = \begin{pmatrix} 3 & 2 \\ 8 & 6 \end{pmatrix} + \begin{pmatrix} 5 & 2k \\ 4 & k \end{pmatrix} = \begin{pmatrix} 8 & 2+2k \\ 12 & 6+k \end{pmatrix} = E$

no Inverse $\Rightarrow \det(E) = 0$

$\Rightarrow \quad 8(6+k) - 12(2+2k) = 0$

$\Rightarrow \quad 48 + 8k - 24 - 24k = 0$

$\Rightarrow \quad 24 = 16k \quad \Rightarrow \quad k = \frac{3}{2}$

3.

$$f(x) = x^2 + \frac{3}{4\sqrt{x}} - 3x - 7, \quad x > 0$$

A root α of the equation $f(x) = 0$ lies in the interval $[3, 5]$.

Taking 4 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 2 decimal places.

(6)

$x_0 = 4 \qquad x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$f(4) = 16 + \frac{3}{8} - 12 - 7 = -\frac{21}{8}$

$f(x) = x^2 + \frac{3}{4}x^{-\frac{1}{2}} - 3x - 7$

$f'(x) = 2x - \frac{3}{8}x^{-\frac{3}{2}} - 3$

$f'(4) = 8 - \frac{3}{64} - 3 = \frac{317}{64}$

$\therefore x_1 = 4 - \dfrac{\frac{-21}{8}}{\frac{317}{64}} = \dfrac{1436}{317} = \underline{4.53}$

4. (a) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to show that

$$\sum_{r=1}^{n}\left(r^3 + 6r - 3\right) = \frac{1}{4}n^2(n^2 + 2n + 13)$$

for all positive integers n.

(5)

(b) Hence find the exact value of

$$\sum_{r=16}^{30}\left(r^3 + 6r - 3\right)$$

(2)

$\sum(r^3+6r-3) = \sum r^3 + 6\sum r - 3\sum 1$

$= \frac{1}{4}n^2(n+1)^2 + \frac{6}{2}n(n+1) - 3n$

$= \frac{1}{4}n\left[n(n+1)^2 + 12(n+1) - 12\right]$

$= \frac{1}{4}n\left[n(n+1)^2 + 12n\right]$

$= \frac{1}{4}n^2\left[(n+1)^2 + 12\right]$

$= \frac{1}{4}n^2\left[n^2+2n+13\right]$ #

b) $\sum_{16}^{30}(r^3+6r-3) = \left[\frac{1}{4}n^2\left[n^2+2n+13\right]\right]_{15}^{30}$

$= (218925) - (15075)$

$= 203850$

5.

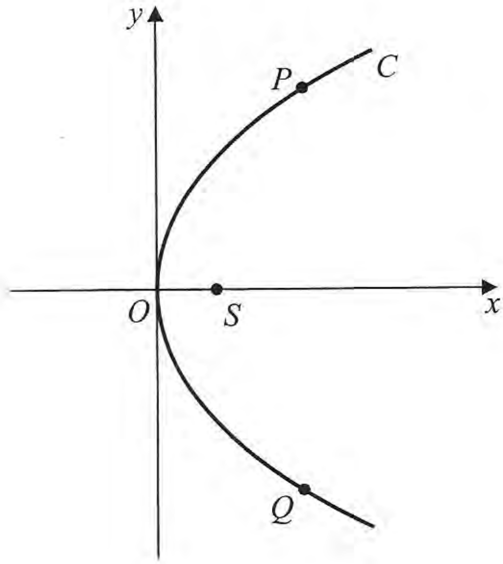


Figure 1

Figure 1 shows a sketch of the parabola C with equation $y^2 = 8x$.
The point P lies on C, where $y > 0$, and the point Q lies on C, where $y < 0$
The line segment PQ is parallel to the y-axis.

Given that the distance PQ is 12,

(a) write down the y-coordinate of P,

(1)

(b) find the x-coordinate of P.

(2)

Figure 1 shows the point S which is the focus of C.
The line l passes through the point P and the point S.

(c) Find an equation for l in the form $ax + by + c = 0$, where a, b and c are integers.

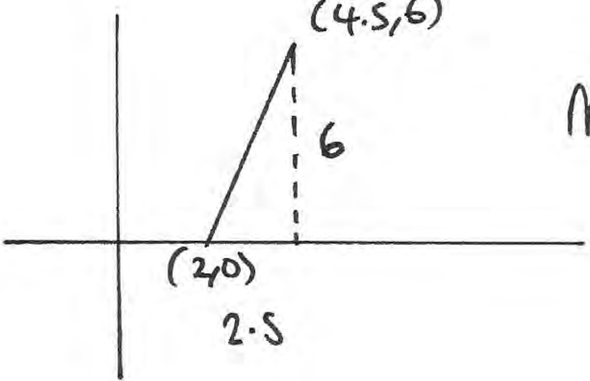
(4)

$y^2 = 4ax \quad \therefore a = 2 \quad \therefore$ focus $(2,0)$
directrix $x+2=0$

$\therefore \; x = 4t^2 \quad y = 4t$

a) $PQ = 12 \Rightarrow$ y coordinate of $P = 6$

b) $y^2 = 8x \Rightarrow \quad 36 = 8x \Rightarrow x = 4.5$



$m = \frac{6}{2.5}$

$y - y_1 = m(x - x_1)$

$y = \frac{6}{2.5}(x-2)$

$2.5y = 6x - 12$

$\underline{12x - 5y - 24 = 0}$

6.

$$f(x) = \tan\left(\frac{x}{2}\right) + 3x - 6, \quad -\pi < x < \pi$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval $[1, 2]$.

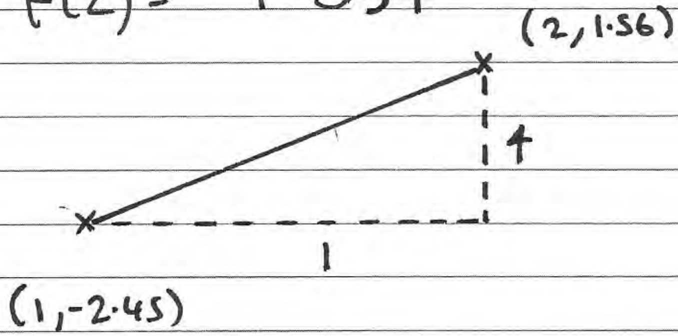
(2)

(b) Use linear interpolation once on the interval $[1, 2]$ to find an approximation to α. Give your answer to 2 decimal places.

(3)

$f(1) = -2.454$ $\therefore$ by change of sign rule

$f(2) = 1.557$ $\alpha \in (1,2)$



$m = \frac{4}{1} = 4$

$y - y_1 = m(x - x_1)$

$y - 1.557 = 4(x - 2)$

$\Rightarrow$ when $y = 0$ $\quad -1.557 = 4(\alpha - 2)$

$-0.389 = \alpha - 2$

$\alpha = 1.61$

7. $$z = 2 - i\sqrt{3}$$

(a) Calculate $\arg z$, giving your answer in radians to 2 decimal places. **(2)**

Use algebra to express

(b) $z + z^2$ in the form $a + bi\sqrt{3}$, where a and b are integers, **(3)**

(c) $\dfrac{z+7}{z-1}$ in the form $c + di\sqrt{3}$, where c and d are integers. **(4)**

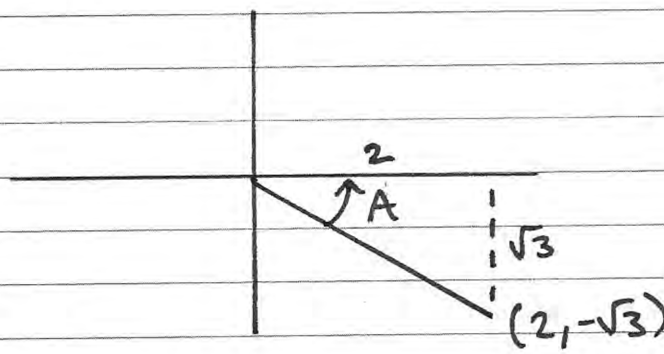
Given that

$$w = \lambda - 3i$$

where λ is a real constant, and $\arg(4 - 5i + 3w) = -\dfrac{\pi}{2}$,

(d) find the value of λ. **(2)**

a)



$A = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$A = 0.714$

$\arg z = \underline{-0.71}$

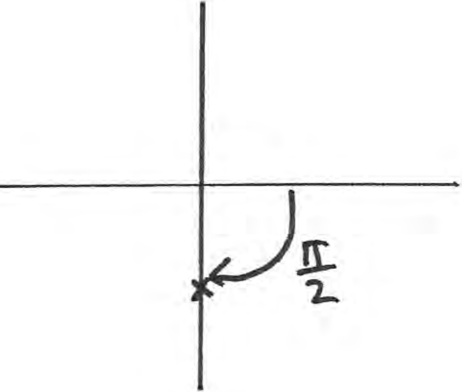
b) $(2-i\sqrt{3}) + (2-i\sqrt{3})^2$

$= (2-i\sqrt{3}) + (1 - i4\sqrt{3})$

$= 3 - i5\sqrt{3}$

c) $\dfrac{z+7}{z-1} = \dfrac{9-i\sqrt{3}}{1-i\sqrt{3}} \times \dfrac{(1+i\sqrt{3})}{(1+i\sqrt{3})} = \dfrac{12+i8\sqrt{3}}{4}$

$= \underline{3 + i2\sqrt{3}}$



$$4-5i+3\omega$$

$$4-5i+3(\lambda-3i)$$

$$(4+3\lambda)-14i$$

$$\therefore\ 4+3\lambda=0$$

$$\lambda=-\frac{4}{3}$$

8. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant.

The point $P\left(ct, \frac{c}{t}\right)$, $t \neq 0$, is a general point on H.

(a) Show that an equation for the tangent to H at P is

$$x + t^2 y = 2ct$$

(4)

The tangent to H at the point P meets the x-axis at the point A and the y-axis at the point B.

Given that the area of the triangle OAB, where O is the origin, is 36,

(b) find the exact value of c, expressing your answer in the form $k\sqrt{2}$, where k is an integer.

(4)

8. $xy = c^2 \Rightarrow y = c^2x^{-1}$

$$\frac{dy}{dx} = -c^2x^{-2} = -\left(\frac{c}{x}\right)^2$$

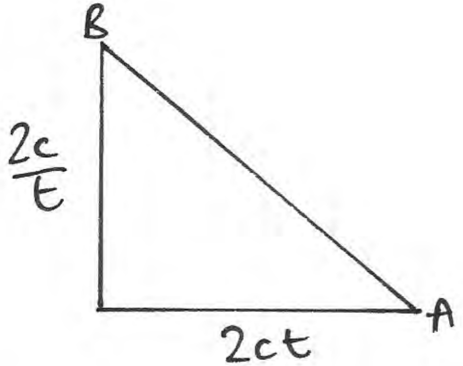
$$P\left(ct, \frac{c}{t}\right) \Rightarrow m_t = -\left(\frac{c}{ct}\right)^2 = -\left(\frac{1}{t}\right)^2 = -\frac{1}{t^2}$$

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct) \Rightarrow t^2y - ct = -x + ct$$

$$\Rightarrow x + t^2y = 2ct \quad \#$$

b) $x = 0 \Rightarrow y = \frac{2ct}{t^2} \Rightarrow \left(0, \frac{2c}{t}\right) B$

$y = 0 \Rightarrow x = 2ct \Rightarrow (2ct, 0) A$



$$\text{Area} = \frac{\cancel{2}ct\cancel{t}\left(\frac{2c}{\cancel{t}}\right)}{\cancel{2}} = 36$$

$\therefore 2c^2 = 36 \quad \therefore c^2 = 18$

$c = \underline{3\sqrt{2}}$

9.

$$\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}$$

(a) Find det **M**.

(1)

The transformation represented by **M** maps the point $S(2a - 7, a - 1)$, where a is a constant, onto the point $S'(25, -14)$.

(b) Find the value of a.

(3)

The point R has coordinates $(6, 0)$.

Given that O is the origin,

(c) find the area of triangle ORS.

(2)

Triangle ORS is mapped onto triangle $OR'S'$ by the transformation represented by **M**.

(d) Find the area of triangle $OR'S'$.

(2)

Given that

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(e) describe fully the single geometrical transformation represented by **A**.

(2)

The transformation represented by **A** followed by the transformation represented by **B** is equivalent to the transformation represented by **M**.

(f) Find **B**.

(4)

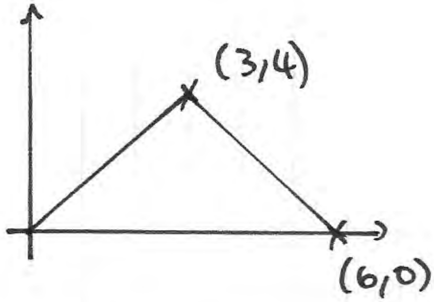
a) $\det M = -15 - 8 = -23$

b) $\begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 2a-7 \\ a-1 \end{pmatrix} = \begin{pmatrix} 25 \\ -14 \end{pmatrix}$

$\therefore \; 3(2a-7)+4(a-1) = 25$

$6a - 21 + 4a - 4 = 25 \;\Rightarrow\; 10a = 50 \;\Rightarrow\; \underline{a = 5}$

c)

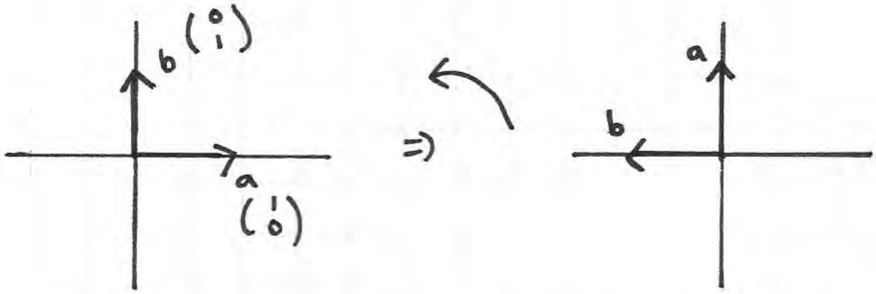


$a = 5 \Rightarrow S(3,4)$

$\text{Area} = \dfrac{6 \times 4}{2} = 12$

d) $\det(M) = -23 \quad \therefore$ AREA OF $S' = 12 \times 23 = 276$

e)



rotation $90°$ anticlockwise about $(0,0)$

f) $BA = M \Rightarrow BAA^{-1} = MA^{-1}$

$\Rightarrow B = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix} A^{-1} \qquad \det(A) = 1 \Rightarrow A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$B = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 5 & 2 \end{pmatrix}$

10. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 2^{2n-1} + 3^{2n-1} \text{ is divisible by } 5.$$

(6)

$n=1 \quad f(1) = 2^1 + 3^1 = 5$ true for $n=1$

$n=2 \quad f(2) = 2^3 + 3^3 = 35 = 7 \times 5$ true for $n=2$

assume true for $n=k$

$\therefore \; f(k) = 2^{2k-1} + 3^{2k-1}$ is divisible by 5

$n=k+1$

$f(k+1) = 2^{2(k+1)-1} + 3^{2(k+1)-1} = 2^{2k+1} + 3^{2k+1}$

$= 2^{(2k-1)+2} + 3^{(2k-1)+2}$

$= 2^2(2^{2k-1}) + 3^2(3^{2k-1})$

$= 4[2^{2k-1} + 3^{2k-1}] + 5(3^{2k-1})$

$= 4f(k) + 5(3^{2k-1})$

Therefore, if $f(k)$ is divisible by 5 $f(k+1)$ must also be divisible by 5

$\therefore$ true for $n=1, n=2$
true for $n=k+1$ if true for $n=k$

$\therefore$ by Induction true for all $n \geq 1$ $n \in \mathbb{Z}^+$