JAN12

1. Given that $z_1 = 1 - \mathrm{i}$,

(a) find $\arg(z_1)$.

(2)

Given also that $z_2 = 3 + 4\mathrm{i}$, find, in the form $a + \mathrm{i}b$, $a, b \in \mathbb{R}$,

(b) $z_1 z_2$,

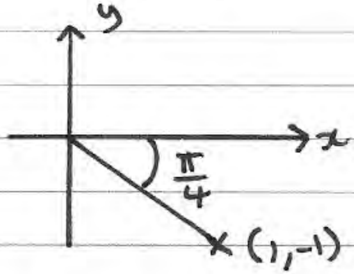
(2)

(c) $\dfrac{z_2}{z_1}$.

(3)

In part (b) and part (c) you must show all your working clearly.

a)



$\therefore \arg(z_1) = -\frac{\pi}{4}$

b) $Z_1 Z_2 = (1-i)(3+4i) = 3 + 4i - 3i - 4i^2$
$= \underline{7 + i}$

c) $\dfrac{Z_2}{Z_1} = \dfrac{3+4i}{1-i}\dfrac{(1+i)}{(1+i)} = \dfrac{3+4i+3i+4i^2}{1-i^2}$

$= \dfrac{-1+7i}{2} = -\frac{1}{2} + \frac{7}{2}i$

2. (a) Show that $f(x) = x^4 + x - 1$ has a real root α in the interval $[0.5, 1.0]$.

(2)

(b) Starting with the interval $[0.5, 1.0]$, use interval bisection twice to find an interval of width 0.125 which contains α.

(3)

(c) Taking 0.75 as a first approximation, apply the Newton Raphson process twice to $f(x)$ to obtain an approximate value of α. Give your answer to 3 decimal places.

(5)

a) $f(0.5) = -0.4375$ } $\therefore$ sign change
$f(1.0) = 1$ } $\alpha \in [0.5, 1.0]$

b)

x	0.5	0.625	0.75	1.0
$f(x)$	-0.4375	-0.222	0.0664	1

$\therefore \quad \alpha \in [0.625, 0.750]$

c) $f'(x) = 4x^3 + 1$

$x_0 = 0.75$

$x_1 = 0.75 - \dfrac{f(0.75)}{f'(0.75)} = 0.7252906977$

$x_2 = 0.725\ldots - \dfrac{f(0.725\ldots)}{f'(0.725\ldots)} = 0.724 \text{ (3dp)}$

3. A parabola C has cartesian equation $y^2 = 16x$. The point $P(4t^2, 8t)$ is a general point on C.

(a) Write down the coordinates of the focus F and the equation of the directrix of C. **(3)**

(b) Show that the equation of the normal to C at P is $y + tx = 8t + 4t^3$. **(5)**

a) $y^2 = 16x = 4ax \Rightarrow a = 4$

focus $(4, 0)$ directrix $x = -4$

b) $\frac{d}{dx} y^2 = \frac{d}{dx} 16x \Rightarrow 2y \frac{dy}{dx} = 16$

$\Rightarrow \frac{dy}{dx} = \frac{16}{2y} = \frac{8}{y}$ M_t at $P = \frac{8}{8t} = \frac{1}{t}$

$\therefore M_n$ at $P = -t$

$y - y_1 = m(x - x_1)$

$y - 8t = -t(x - 4t^2)$

$y - 8t = -tx + 4t^3$

$\therefore y + tx = 8t + 4t^3$ #

4. A right angled triangle T has vertices $A(1,1)$, $B(2,1)$ and $C(2,4)$. When T is transformed by the matrix $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the image is T'.

(a) Find the coordinates of the vertices of T'.

(2)

(b) Describe fully the transformation represented by $\mathbf{P}$.

(2)

The matrices $\mathbf{Q} = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ represent two transformations. When T is transformed by the matrix $\mathbf{QR}$, the image is T''.

(c) Find $\mathbf{QR}$.

(2)

(d) Find the determinant of $\mathbf{QR}$.

(2)

(e) Using your answer to part (d), find the area of T''.

(3)

a) $T' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \end{pmatrix}$

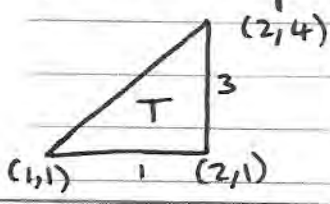
$A'(1,1) \quad B'(1,2) \quad C'(4,2)$

b) reflection through $y=x$

c) $QR = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

d) $\det QR = (-2)(2) - (0)(0) = -4$

e) Area of $T'' = 4 \times$ Area of T



Area of $T = \frac{3}{2}$ $\quad \therefore$ Area of $T'' = 6$

5. The roots of the equation

$$z^3 - 8z^2 + 22z - 20 = 0$$

are z_1, z_2 and z_3.

(a) Given that $z_1 = 3 + \mathrm{i}$, find z_2 and z_3.

(4)

(b) Show, on a single Argand diagram, the points representing z_1, z_2 and z_3.

(2)

a) $Z_1 = 3+i \Rightarrow Z_2 = 3-i$

Sum of roots $= 6$

product of roots $= (3+i)(3-i) = 10$

$\therefore\ x^2 - (z_1 + z_2)x + (z_1 z_2) = 0$

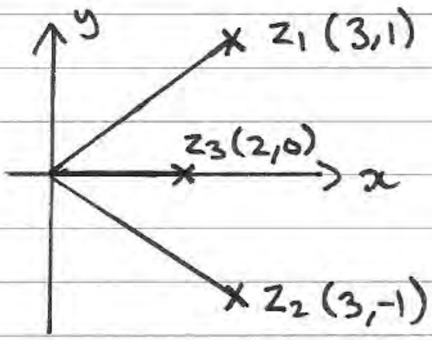
$\Rightarrow x^2 - 6x + 10$

$(z^2 - 6z + 10)(az + b)$

$\therefore\ a = 1 \quad b = -2 \qquad (z^2 - 6z + 10)(z - 2) = 0$

$z_1 = 3+i \qquad z_2 = 3 - i \qquad z_3 = 2$

b)



6. (a) Prove by induction

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(5)

(b) Using the result in part (a), show that

$$\sum_{r=1}^{n} (r^3 - 2) = \frac{1}{4}n(n^3 + 2n^2 + n - 8)$$

(3)

(c) Calculate the exact value of $\sum_{r=20}^{50} (r^3 - 2)$.

(3)

$n=1 \quad \sum_{1}^{1} r^3 = 1^3 = 1$

$\frac{1}{4}n^2(n+1)^2 = \frac{1}{4}(1)^2(2)^2 = 1$

$\therefore$ true for $n=1$

assume true for $n=k$

$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$

$n=k+1 \quad \sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2(k+1+1)^2$

$= \frac{1}{4}(k+1)^2(k+2)^2$

$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$

$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$

$= (k+1)^2\left[\frac{1}{4}k^2 + (k+1)\right]$

$= \frac{1}{4}(k+1)^2 [k^2 + 4k + 4]$

$= \frac{1}{4}(k+1)^2 (k+2)^2$ #

$\therefore$ true if $n=1$
true for $n=k+1$, if true for $n=k$
$\therefore$ true for all $n \in \mathbb{Z}$ by induction.

b) $\sum_{r=1}^{n} r^3 - 2 = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} 2$

$= \frac{1}{4}n^2(n+1)^2 - 2n$

$= \frac{1}{4}n[n(n+1)^2 - 8]$

$= \frac{1}{4}n[n(n^2+2n+1) - 8]$

$= \frac{1}{4}n[n^3 + 2n^2 + n - 8]$ #

c) $\sum_{r=20}^{50} (r^3 - 2) = \sum_{r=1}^{50} (r^3 - 2) - \sum_{r=1}^{19} (r^3 - 2)$

$= \frac{1}{4}(50)[50^3 + 2(50)^2 + (50) - 8]$
$- \frac{1}{4}(19)[19^3 + 2(19)^2 + (19) - 8]$

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7. A sequence can be described by the recurrence formula

$$u_{n+1} = 2u_n + 1, \qquad n \geqslant 1, \quad u_1 = 1$$

(a) Find u_2 and u_3.

(2)

(b) Prove by induction that $u_n = 2^n - 1$

(5)

a) $U_1 = 1$

$U_2 = 2U_1 + 1 = 2 \times 1 + 1 = 3$

$U_3 = 2U_2 + 1 = 2 \times 3 + 1 = 7$

b) $n=1 \quad U_1 = 1 \quad U_1 = 2^1 - 1 = 1$ ✓

$n=2 \quad U_2 = 3 \quad U_2 = 2^2 - 1 = 3$ ✓

$n=3 \quad U_3 = 7 \quad U_3 = 2^3 - 1 = 7$ ✓

$\therefore$ true for $n=1, n=2, n=3$

assume true for $n=k \therefore 2^k - 1 = U_k$

$n=k+1 \qquad U_{k+1} = 2^{k+1} - 1$

$U_{k+1} = 2U_k + 1 = 2[2^k - 1] + 1$

$= 2^{k+1} - 2 + 1$

$= 2^{k+1} - 1$ #

$\therefore$ true for $n=1, n=2$

if true for $n=k$, true for $n=k+1$

$\therefore$ by induction true for all $n \in \mathbb{Z}$

8.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

(a) Show that $\mathbf{A}$ is non-singular.

(2)

(b) Find $\mathbf{B}$ such that $\mathbf{BA}^2 = \mathbf{A}$.

(4)

a) $\det A = 0 \times 3 - 1 \times 2 = -2$

$\det A \neq 0 \Rightarrow A$ is non-singular

b) $BA^2 = A \Rightarrow BA^2A^{-1} = AA^{-1} = I$

$\Rightarrow BA^2A^{-1}A^{-1} = A^{-1}$

$\therefore B = A^{-1}$

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 3 & -1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

9. The rectangular hyperbola H has cartesian equation $xy = 9$

The points $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ lie on H, where $p \neq \pm q$.

(a) Show that the equation of the tangent at P is $x + p^2 y = 6p$.

(4)

(b) Write down the equation of the tangent at Q.

(1)

The tangent at the point P and the tangent at the point Q intersect at R.

(c) Find, as single fractions in their simplest form, the coordinates of R in terms of p and q.

(4)

$xy = 9 = c^2 \quad \therefore c^2 = 9$

$y = \frac{9}{x} \Rightarrow y = 9x^{-1}$

$\therefore \frac{dy}{dx} = -9x^{-2} = \frac{-9}{x^2}$

$\Rightarrow m_t \text{ at } P = \frac{-9}{(3p)^2} = -\frac{1}{p^2} \quad P\left(3p, \frac{3}{p}\right)$

$y - y_1 = m(x - x_1) \Rightarrow y - \frac{3}{p} = -\frac{1}{p^2}(x - 3p)$

$\times p^2 \Rightarrow p^2 y - 3p = -x + 3p$

$\therefore x + p^2 y = 6p$ #

b) $x + q^2 y = 6q$

c) $x = 6p - p^2 y \; ; \; x = 6q - q^2 y$

at $R \Rightarrow 6p - p^2 y = 6q - q^2 y$

$q^2 y - p^2 y = 6q - 6p$

$$\therefore\ y(q^2 - p^2) = 6q - 6p$$

$$\Rightarrow\ y = \frac{6\cancel{(q-p)}}{\cancel{(q-p)}(q+p)} = \frac{6}{p+q}.$$

$$x = 6p - p^2 y = 6p - p^2\left(\frac{6}{p+q}\right)$$

$$x = \frac{6p(p+q) - 6p^2}{p+q} = \frac{\cancel{6p^2} + 6pq - \cancel{6p^2}}{p+q}$$

$$x = \frac{6pq}{p+q} \qquad R\left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)$$