

FPI JAN 11

1.

$$z = 5 - 3i, \quad w = 2 + 2i$$

Express in the form $a + bi$, where a and b are real constants,

(a) z^2 ,

(2)

(b) $\frac{z}{w}$.

(3)

$$a) \quad z^2 = (5-3i)^2 = 25 - 30i + 9i^2 = 16 - 30i$$

$$b) \quad \frac{5-3i}{2+2i} \times \frac{2-2i}{2-2i} = \frac{10-16i+6i^2}{4-4i^2} = \frac{4-16i}{8} = \frac{1}{2} - 2i$$

2.

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$$

(a) Find \mathbf{AB} .

(3)

Given that

$$\mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

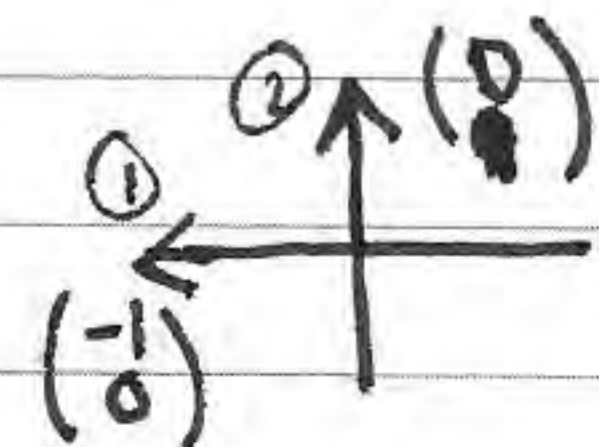
(b) describe fully the geometrical transformation represented by \mathbf{C} ,

(2)

(c) write down \mathbf{C}^{100} .

(1)

$$a) \quad \mathbf{AB} = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -6 & -2 \\ 0 & 1 \end{pmatrix}$$

$$b) \quad \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$


reflected through y-axis

$$c) \quad \mathbf{C}^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.

$$f(x) = 5x^2 - 4x^{\frac{3}{2}} - 6, \quad x \geq 0$$

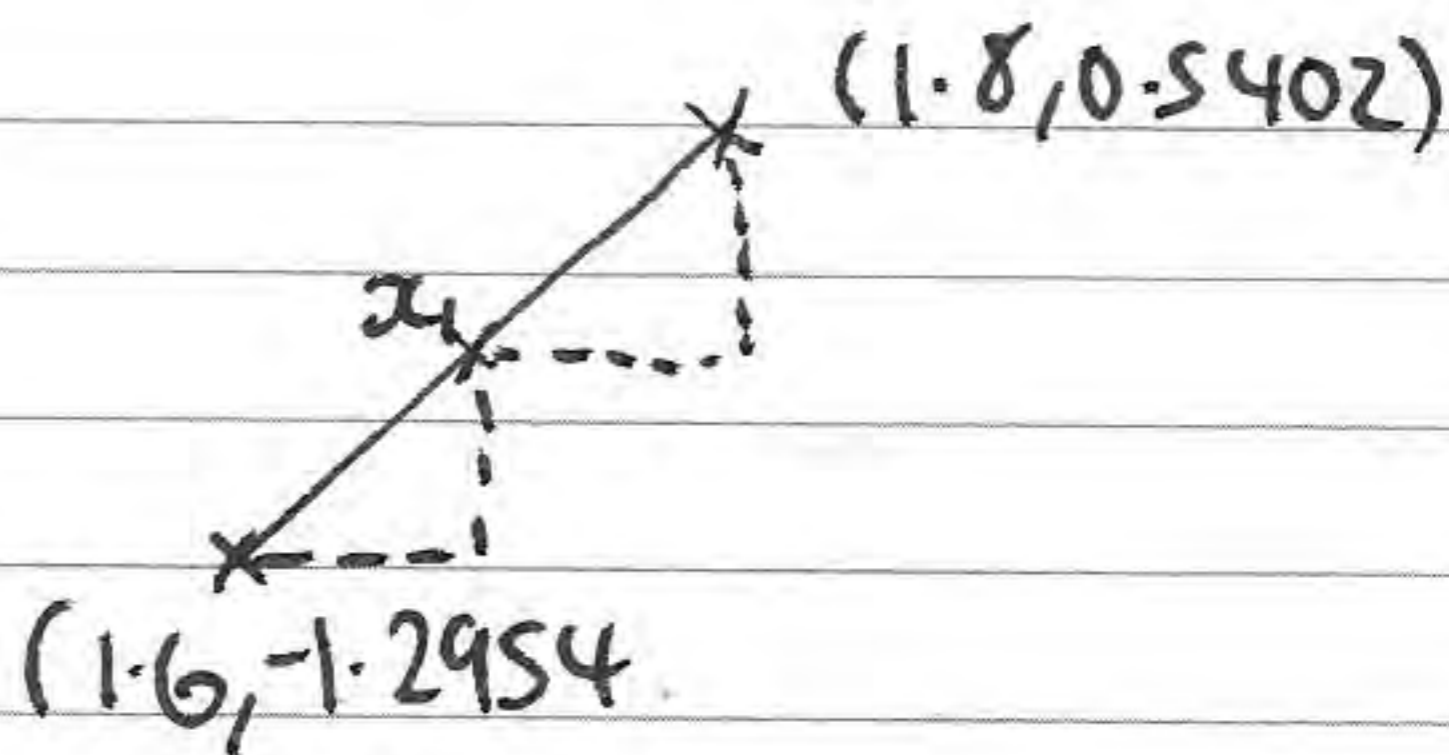
The root α of the equation $f(x) = 0$ lies in the interval $[1.6, 1.8]$.

(a) Use linear interpolation once on the interval $[1.6, 1.8]$ to find an approximation to α . Give your answer to 3 decimal places. (4)

(b) Differentiate $f(x)$ to find $f'(x)$. (2)

(c) Taking 1.7 as a first approximation to α , apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α . Give your answer to 3 decimal places. (4)

$$\begin{aligned} \text{a) } f(1.6) &= -1.2954 \\ f(1.8) &= 0.5402 \end{aligned}$$



$$\frac{1.8 - x}{x - 1.6} = \frac{0.5402}{1.2954}$$

$$1.8 - x = 0.41699x - 0.66719$$

$$1.41699x = 2.46719 \quad \Rightarrow \quad x = \underline{1.741}$$

$$\text{b) } f'(x) = 10x - 6x^{\frac{1}{2}}$$

$$\text{c) } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 1.7 - \frac{5(1.7)^2 - 4(1.7)^{\frac{3}{2}} - 6}{10(1.7) - 6(1.7)^{\frac{1}{2}}}$$

$$x_0 = 1.7, \quad x_1 = 1.745$$

4. Given that $2 - 4i$ is a root of the equation

$$z^2 + pz + q = 0,$$

where p and q are real constants,

(a) write down the other root of the equation,

(1)

(b) find the value of p and the value of q .

(3)

a) $2+4i$

b) $\alpha = 2-4i$

$\beta = 2+4i$

$\alpha + \beta = 4 = -p$

$\alpha\beta = 4 - 16i^2 = 20 = q$

$p = 4$

$q = 20$

5. (a) Use the results for $\sum_{r=1}^n r$, $\sum_{r=1}^n r^2$ and $\sum_{r=1}^n r^3$, to prove that

$$\sum_{r=1}^n r(r+1)(r+5) = \frac{1}{4}n(n+1)(n+2)(n+7)$$

for all positive integers n .

(5)

- (b) Hence, or otherwise, find the value of

$$\sum_{r=20}^{50} r(r+1)(r+5)$$

(2)

a) $\sum r = \frac{1}{2}n(n+1)$

$$\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum r^3 = \frac{1}{4}n^2(n+1)^2$$

$$\begin{aligned} & r(r+1)(r+5) \\ & (r^2+r)(r+5) \\ & (r^3+6r^2+5r) \end{aligned}$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{4} \times 4n(n+1)(2n+1) + \frac{1}{4} \times 10n(n+1)$$

$$= \frac{1}{4}n(n+1)[n(n+1) + 4(2n+1) + 10]$$

$$= \frac{1}{4}n(n+1)[n^2 + 9n + 14] = \frac{1}{4}n(n+1)(n+2)(n+7) \quad \#$$

b)

$$\left[\frac{1}{4}n(n+1)(n+2)(n+7) \right]_{19}^{50} = \left(\frac{1}{4}(50)(51)(52)(57) \right) - \left(\frac{1}{4}(19)(20)(21)(26) \right)$$

$$= \underline{1837680}$$

6.

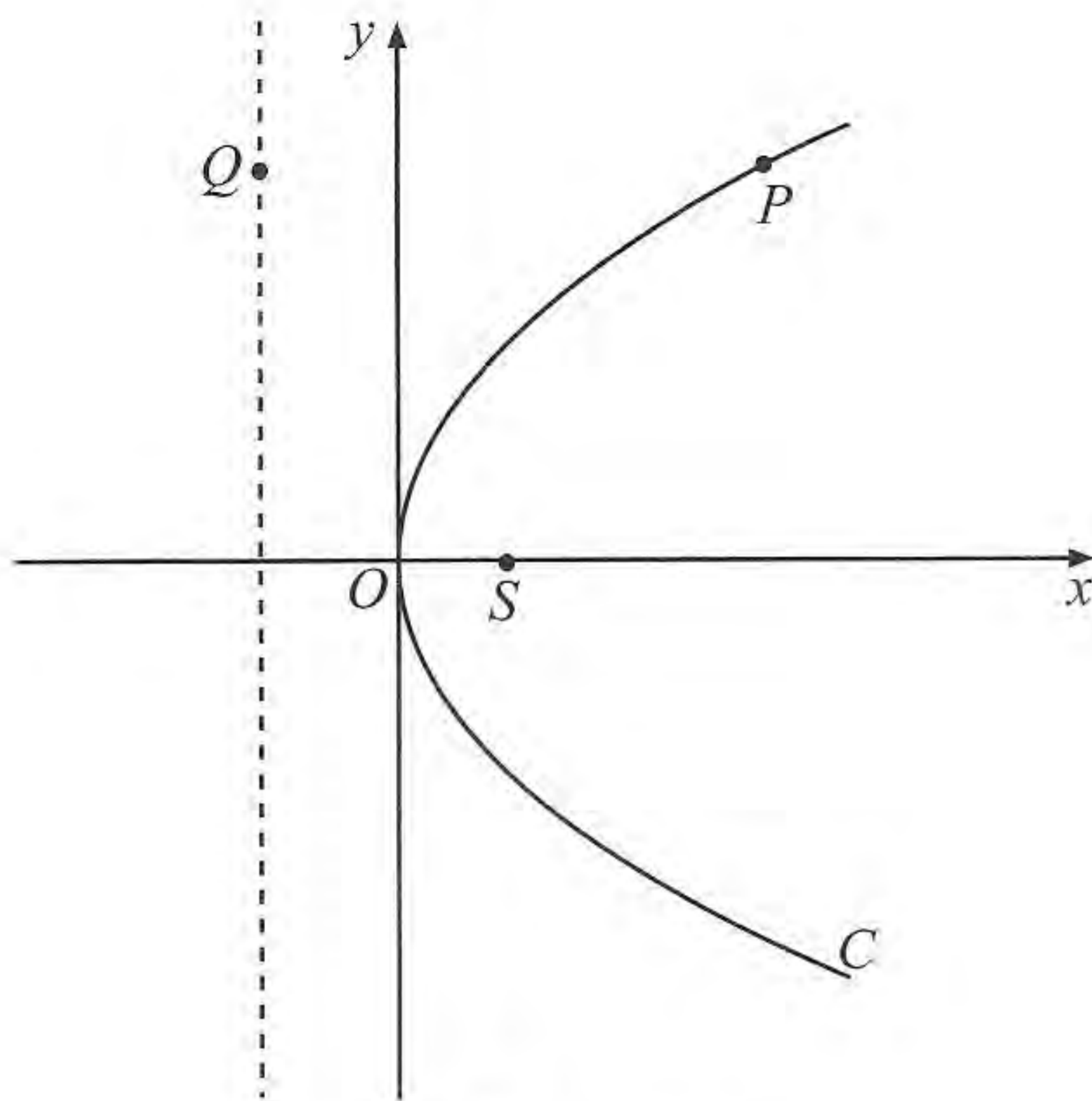


Figure 1

Figure 1 shows a sketch of the parabola C with equation $y^2 = 36x = 4ax$ $a=9$. The point S is the focus of C .

(a) Find the coordinates of S . $(9,0)$ (1)

(b) Write down the equation of the directrix of C . $x+9=0$ $x=-9$ (1)

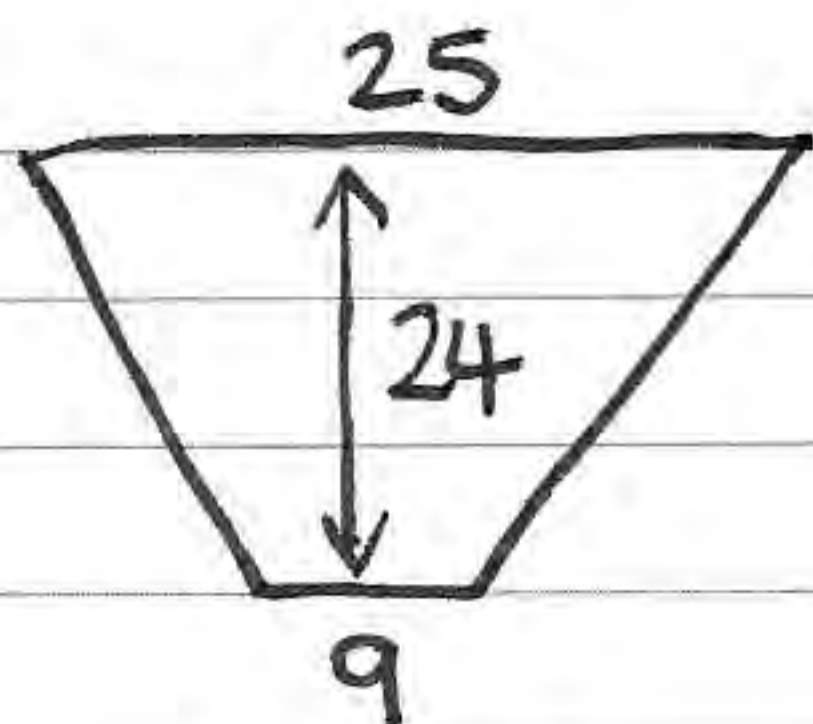
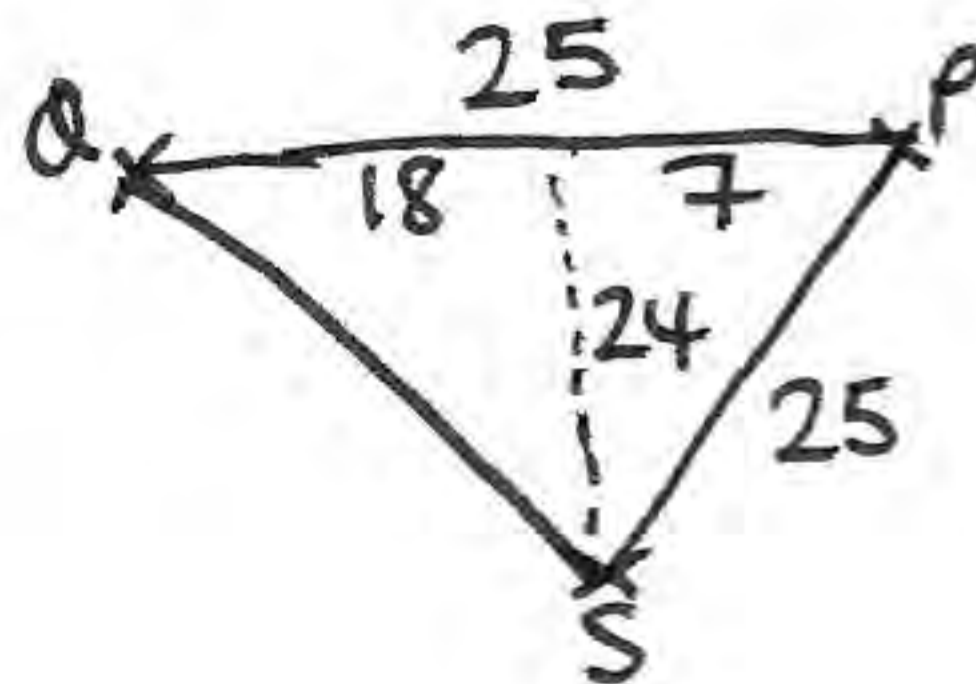
Figure 1 shows the point P which lies on C , where $y > 0$, and the point Q which lies on the directrix of C . The line segment QP is parallel to the x -axis.

Given that the distance PS is 25,

(c) write down the distance QP , $= 25$ (1)

(d) find the coordinates of P , $(16,24)$ (3)

(e) find the area of the trapezium $OSPQ$. (2)



$$\frac{34 \times 24}{2} = 408$$

7.

$$z = -24 - 7i$$

(a) Show z on an Argand diagram.

(1)

(b) Calculate $\arg z$, giving your answer in radians to 2 decimal places.

(2)

It is given that

$$w = a + bi, \quad a \in \mathbb{R}, b \in \mathbb{R}$$

Given also that $|w| = 4$ and $\arg w = \frac{5\pi}{6}$,

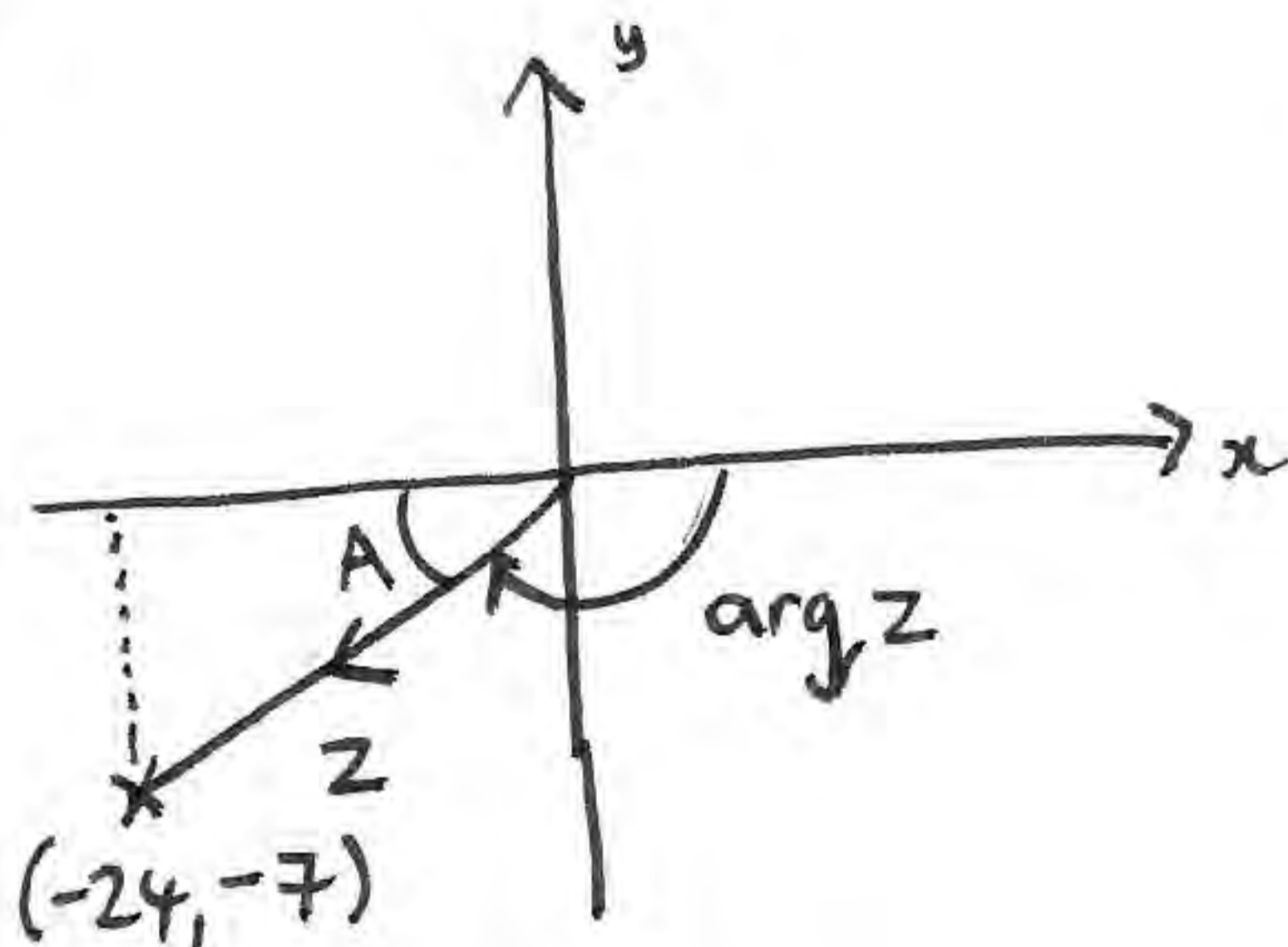
(c) find the values of a and b ,

(3)

(d) find the value of $|zw|$.

(3)

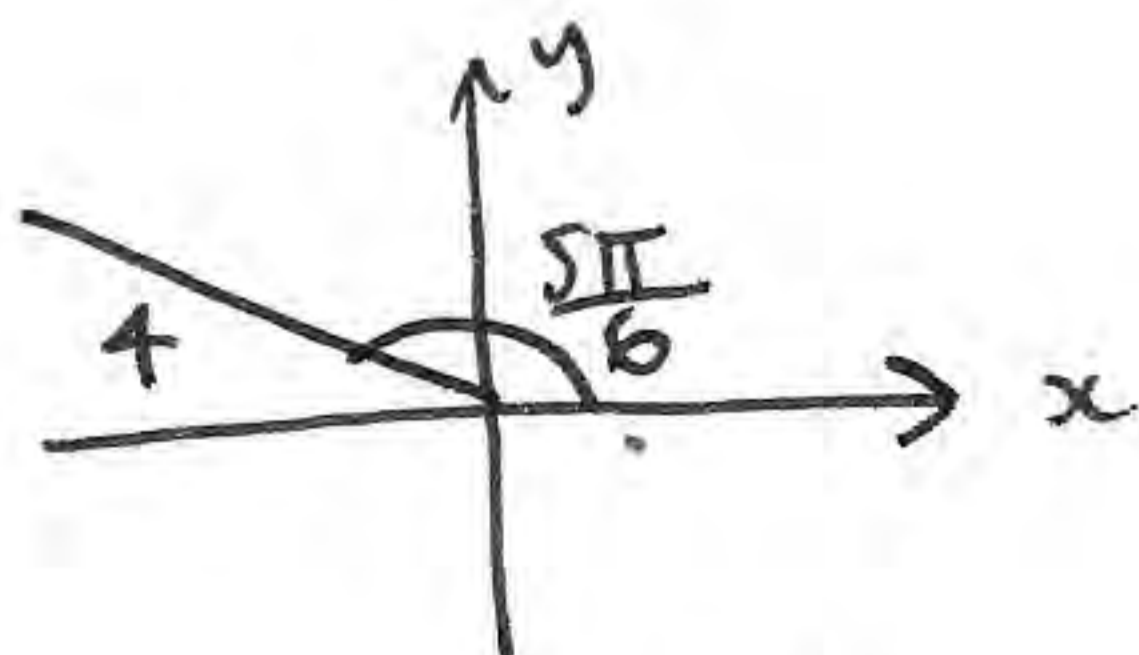
a)



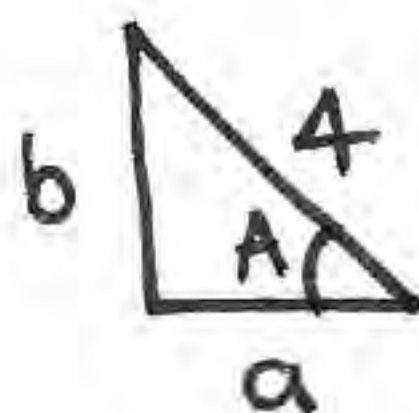
$$b) \arg z = \left(\pi - \tan^{-1} \left(\frac{7}{24} \right) \right)$$

$$\arg z = -2.86^{\circ}$$

c)



$$w = -2\sqrt{3} + 2i$$



$$A = \frac{\pi}{6}$$

$$a = 4 \cos \frac{\pi}{6} = 2\sqrt{3}$$

$$b = 4 \sin \frac{\pi}{6} = 2$$

$$d) |zw| = |z||w| = 25 \times 4 = \underline{\underline{100}}$$

8.

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

(a) Find $\det A$.

(1)

(b) Find A^{-1} .

(2)

The triangle R is transformed to the triangle S by the matrix A .
Given that the area of triangle S is 72 square units,

(c) find the area of triangle R .

(2)

The triangle S has vertices at the points $(0, 4)$, $(8, 16)$ and $(12, 4)$.

(d) Find the coordinates of the vertices of R .

(4)

$$a) \det A = 2 \times 3 - (-2)(-1) = 6 - 2 = 4$$

$$b) A^{-1} = \frac{1}{4} \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$c) R \times \det A = S$$

$$\text{area } R = \frac{72}{4} = 18$$

$$d) \begin{pmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 8 & 12 \\ 4 & 16 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 14 & 11 \\ 2 & 10 & 5 \end{pmatrix}$$

$$(2, 2) ; (14, 10) ; (11, 5)$$

9. A sequence of numbers $u_1, u_2, u_3, u_4, \dots$ is defined by

$$u_{n+1} = 4u_n + 2, \quad u_1 = 2$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = \frac{2}{3}(4^n - 1)$$

(5)

$$u_1 = \frac{2}{3}(4^1 - 1) = \frac{2}{3} \times 3 = 2 \quad \checkmark$$

$$u_2 = \frac{2}{3}(4^2 - 1) = \frac{2}{3} \times 15 = 10$$

$$u_2 = 4u_1 + 2 = 4 \times 2 + 2 = 10 \quad \checkmark$$

assume rule works for $n=k$

$$u_k = \frac{2}{3}(4^k - 1)$$

$$u_{k+1} = \frac{2}{3}(4^{k+1} - 1)$$

$$u_{k+1} = 4 \left[\frac{2}{3}(4^k - 1) \right] + 2$$

$$= \frac{8}{3}(4^k - 1) + 2 = \frac{2}{3} [4(4^k - 1) + 3]$$

$$= \frac{2}{3} [4^{k+1} - 4 + 3]$$

$$= \frac{2}{3} [4^{k+1} - 1] \quad \# .$$

rule works for $n=1, n=2$; if it works for $n=k$, it must work for $n=k+1$ so it is true for all $n \geq 1, n \in \mathbb{Z}^+$ by mathematical induction.

10. The point $P\left(6t, \frac{6}{t}\right)$, $t \neq 0$, lies on the rectangular hyperbola H with equation $xy = 36$.

(a) Show that an equation for the tangent to H at P is

$$y = -\frac{1}{t^2}x + \frac{12}{t} \quad (5)$$

The tangent to H at the point A and the tangent to H at the point B meet at the point $(-9, 12)$.

(b) Find the coordinates of A and B .

(7)

$$a) \quad xy = 36 = c^2 \quad x = ct \quad y = \frac{c}{t}$$

$$\Rightarrow c = 6 \quad x = 6t \quad y = \frac{6}{t}$$

$$y = 36x^{-1}$$

$$\frac{dy}{dx} = -36x^{-2} = \frac{-36}{x^2} \quad x = 6t \quad m_t = \frac{-36}{36t^2} = \frac{-1}{t^2}$$

$$y - y_1 = m(x - x_1) \Rightarrow y - \frac{6}{t} = \frac{-1}{t^2}(x - 6t)$$

$$y - \frac{6}{t} = -\frac{1}{t^2}x + \frac{6}{t} \Rightarrow y = -\frac{1}{t^2}x + \frac{12}{t} \quad \neq$$

$$b) \quad 12 = -\frac{1}{t^2}x - 9 + \frac{12}{t} \Rightarrow 12 = \frac{9}{t^2} + \frac{12}{t} \quad (\times t^2)$$

$$12t^2 = 9 + 12t \Rightarrow 4t^2 - 4t - 3 = 0 \Rightarrow (2t+1)(2t-3)$$

$$t = -\frac{1}{2}, t = \frac{3}{2} \quad t = -\frac{1}{2} \quad (\cancel{A(6, 4)}) \quad (-3, -12)$$

$$t = \frac{3}{2} \quad (9, 4)$$