

CASIS

1. (a) Find the binomial expansion of

$$(4 + 5x)^{\frac{1}{2}}, \quad |x| < \frac{4}{5}$$

in ascending powers of x , up to and including the term in x^2 .
Give each coefficient in its simplest form.



(5)

- (b) Find the exact value of $(4 + 5x)^{\frac{1}{2}}$ when $x = \frac{1}{10}$

Give your answer in the form $k\sqrt{2}$, where k is a constant to be determined.

(1)

- (c) Substitute $x = \frac{1}{10}$ into your binomial expansion from part (a) and hence find an approximate value for $\sqrt{2}$

Give your answer in the form $\frac{p}{q}$ where p and q are integers.

(2)

$$a) \quad 4^{\frac{1}{2}} \left(1 + \frac{5}{4}x\right)^{\frac{1}{2}} = 2 \left(1 + \frac{5}{4}x\right)^{\frac{1}{2}}$$

$$= 2 \left[1 + \left(\frac{1}{2}\right)\left(\frac{5}{4}x\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2} \left(\frac{5}{4}x\right)^2 \right] = 2 + \frac{5}{4}x + \frac{-25}{64}x^2$$

$$b) \quad \left(4 + \frac{5}{10}\right)^{\frac{1}{2}} = \left(\frac{9}{2}\right)^{\frac{1}{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \quad \therefore k = \frac{3}{2}$$

$$c) \quad x = \frac{1}{10} \Rightarrow \frac{3}{2}\sqrt{2} \approx 2 + \frac{5}{4}\left(\frac{1}{10}\right) - \frac{25}{64}\left(\frac{1}{10}\right)^2 = \frac{543}{256}$$

$$\left(\frac{x \cdot 2}{3}\right)$$

$$\therefore \sqrt{2} \approx \frac{181}{128}$$

2. The curve C has equation

$$x^2 - 3xy - 4y^2 + 64 = 0$$

(a) Find $\frac{dy}{dx}$ in terms of x and y .

(5)

(b) Find the coordinates of the points on C where $\frac{dy}{dx} = 0$

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(6)

$$a) \frac{d}{dx}(x^2) - \frac{d}{dx}(3xy) - \frac{d}{dx}(4y^2) + \frac{d}{dx}(64) = 0$$

$$\Rightarrow 2x - 3y - 3x \frac{dy}{dx} - 8y \frac{dy}{dx} = 0$$

$$\Rightarrow (3x + 8y) \frac{dy}{dx} = 2x - 3y \quad \therefore \frac{dy}{dx} = \frac{2x - 3y}{3x + 8y}$$

$$b) \frac{dy}{dx} = 0 \Rightarrow 2x - 3y = 0 \Rightarrow 2x = 3y \quad \therefore y = \frac{2}{3}x$$

$$\Rightarrow x^2 - 3x\left(\frac{2}{3}x\right) - 4\left(\frac{2}{3}x\right)^2 + 64 = 0$$

$$\Rightarrow x^2 - 2x^2 - \frac{16}{9}x^2 + 64 = 0$$

$$\textcircled{\times 9} \quad 9x^2 - 18x^2 - 16x^2 + 576 = 0$$

$$\Rightarrow 25x^2 = 576 \Rightarrow x^2 = \frac{576}{25} \Rightarrow x = \pm \frac{24}{5}$$

$$y = \frac{2}{3}x \quad \therefore y = \pm \frac{48}{15} = \pm \frac{16}{5} \quad C \left(\frac{24}{5}, \frac{16}{5} \right); \left(\frac{-24}{5}, \frac{-16}{5} \right)$$

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3.

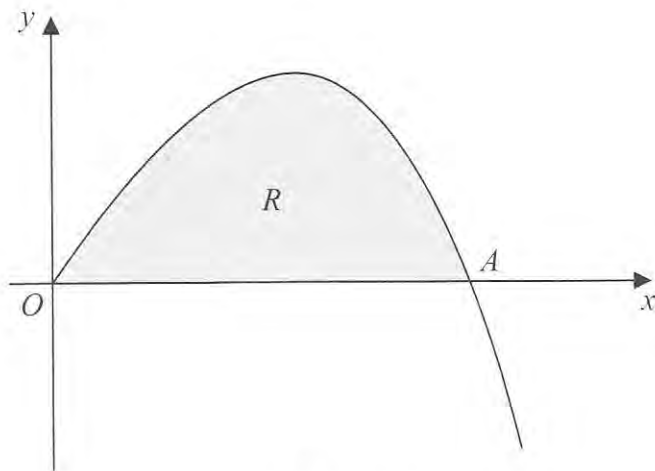


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = 4x - xe^{\frac{1}{2}x}$, $x \geq 0$

The curve meets the x -axis at the origin O and cuts the x -axis at the point A .

(a) Find, in terms of $\ln 2$, the x coordinate of the point A .

(2)

(b) Find

$$\int xe^{\frac{1}{2}x} dx$$

(3)

The finite region R , shown shaded in Figure 1, is bounded by the x -axis and the curve with equation

$$y = 4x - xe^{\frac{1}{2}x}, \quad x \geq 0$$

(c) Find, by integration, the exact value for the area of R .

Give your answer in terms of $\ln 2$

(3)

$$a) y=0 \Rightarrow 4x = x e^{\frac{1}{2}x} \Rightarrow \ln 4 = \frac{1}{2}x \therefore x = 2 \ln 4$$

$$\therefore x = 2 \ln 2^2 = 4 \ln 2$$

$$b) \int x e^{\frac{1}{2}x} dx \quad \int u v' = u v - \int u' v$$

$$u = x \quad v = 2 e^{\frac{1}{2}x}$$

$$u' = 1 \quad v' = e^{\frac{1}{2}x}$$

$$= 2x e^{\frac{1}{2}x} - \int 2 e^{\frac{1}{2}x} dx$$

$$= 2x e^{\frac{1}{2}x} - 4 e^{\frac{1}{2}x} + c \quad (2(x-2)e^{\frac{1}{2}x} + c)$$

$$c) R = \int_0^{4 \ln 2} 4x - x e^{\frac{1}{2}x} dx = \left[2x^2 - 2x e^{\frac{1}{2}x} + 4 e^{\frac{1}{2}x} \right]_0^{4 \ln 2}$$

$$x = 2 \ln 4$$

$$\Rightarrow \frac{1}{2}x = \ln 4$$

$$= \left(32(\ln 2)^2 - 8 \ln 2 e^{\ln 4} + 4 e^{\ln 4} \right) - (0 - 0 + 4)$$

$$= 32(\ln 2)^2 - 32 \ln 2 + 16 - 4$$

$$= 32(\ln 2)^2 - 32 \ln 2 + 12$$

4. With respect to a fixed origin O , the lines l_1 and l_2 are given by the equations

$$l_1: \mathbf{r} = \begin{pmatrix} 5 \\ -3 \\ p \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \quad l_2: \mathbf{r} = \begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$$

where λ and μ are scalar parameters and p is a constant.

The lines l_1 and l_2 intersect at the point A .

- (a) Find the coordinates of A .

(2)

- (b) Find the value of the constant p .

(3)

- (c) Find the acute angle between l_1 and l_2 , giving your answer in degrees to 2 decimal places.

(3)

The point B lies on l_2 where $\mu = 1$

- (d) Find the shortest distance from the point B to the line l_1 , giving your answer to 3 significant figures.

(3)

$$a) \Gamma_1 = \Gamma_2 \Rightarrow \begin{pmatrix} 5 \\ -3+\lambda \\ p-3\lambda \end{pmatrix} = \begin{pmatrix} 8+3\mu \\ 5+4\mu \\ -2-5\mu \end{pmatrix} \quad \textcircled{i} \quad 5 = 8+3\mu \Rightarrow \underline{\underline{\mu = -1}}$$

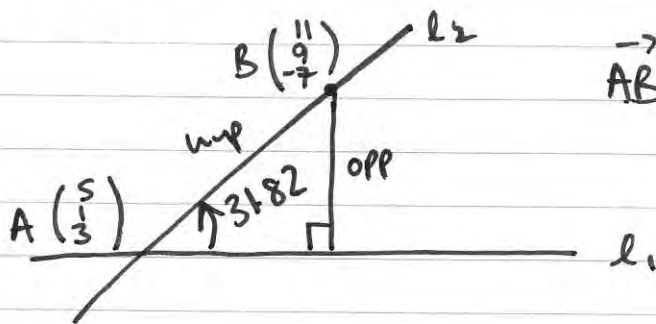
$$\therefore A(8-3, 5-4, -2+5) \quad \therefore A(5, 1, 3)$$

$$b) \textcircled{j} \quad -3+\lambda = 5+4(-1) \Rightarrow -3+\lambda = 1 \quad \therefore \underline{\underline{\lambda = 4}}$$

$$\textcircled{k} \quad p-3(4) = -2-5(-1) \Rightarrow p-12 = -2+5 \quad \therefore \underline{\underline{p = 15}}$$

$$c) \cos \theta = \frac{\left| \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \right|}{\left| \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right| \left| \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \right|} \Rightarrow \cos \theta = \frac{19}{\sqrt{10} \sqrt{50}} \quad \therefore \underline{\underline{\theta = 31.82^\circ}}$$

$$d) B \begin{pmatrix} 8+3 \\ 5+4 \\ -2-5 \end{pmatrix} \quad B \begin{pmatrix} 11 \\ 9 \\ -7 \end{pmatrix} \quad B(11, 9, -7) \text{ on } l_2.$$



$$\vec{AB} = \begin{pmatrix} 11 \\ 9 \\ -7 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ -10 \end{pmatrix}$$

$$\therefore |\vec{AB}| = \sqrt{6^2 + 8^2 + 10^2} \\ = \sqrt{200}$$

$$\therefore \text{Shortest distance} = \sqrt{200} \times \sin 31.82 \\ = \underline{\underline{7.46}}$$

5. A curve C has parametric equations

$$x = 4t + 3, \quad y = 4t + 8 + \frac{5}{2t}, \quad t \neq 0$$

(a) Find the value of $\frac{dy}{dx}$ at the point on C where $t = 2$, giving your answer as a fraction in its simplest form.

(3)

(b) Show that the cartesian equation of the curve C can be written in the form

$$y = \frac{x^2 + ax + b}{x - 3}, \quad x \neq 3$$

where a and b are integers to be determined.

(3)

$$a) \quad \frac{dy}{dt} = 4 - \frac{5}{2t^2} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4 - \frac{5}{2t^2}}{4}$$

$$\frac{dx}{dt} = 4$$

$$\therefore \frac{dy}{dx} = \frac{8t^2 - 5}{8t^2} \quad \text{When } t = 2, \\ \frac{dy}{dx} = \frac{8(2)^2 - 5}{8(2)^2} \\ = \frac{27}{32}$$

$$b) \quad y = 4t + 8 + \frac{10}{4t} \quad 4t = x - 3$$

$$\therefore y = (x - 3) + 8 + \frac{10}{x - 3} = x + 5 + \frac{10}{x - 3}$$

$$\therefore y = \frac{(x + 5)(x - 3) + 10}{x - 3} = \frac{x^2 + 2x - 15 + 10}{x - 3}$$

$$\therefore y = \frac{x^2 + 2x - 5}{x - 3}$$

6.

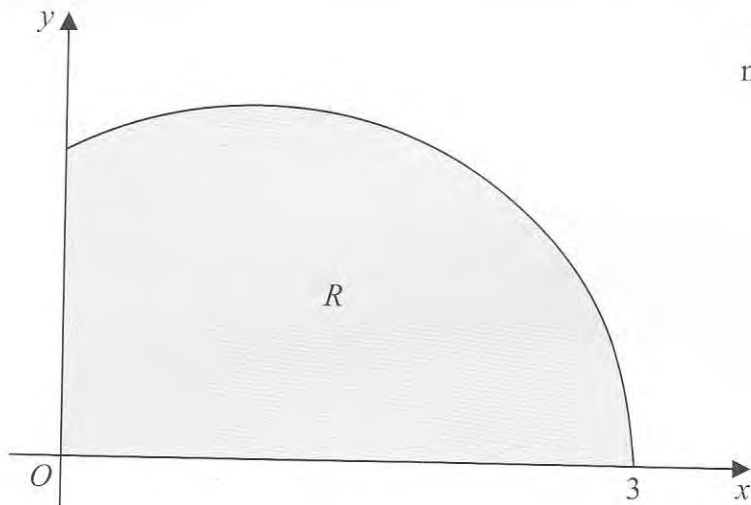
Diagram
not to scale**Figure 2**

Figure 2 shows a sketch of the curve with equation $y = \sqrt{(3-x)(x+1)}$, $0 \leq x \leq 3$

The finite region R , shown shaded in Figure 2, is bounded by the curve, the x -axis, and the y -axis.

(a) Use the substitution $x = 1 + 2 \sin \theta$ to show that

$$\int_0^3 \sqrt{(3-x)(x+1)} \, dx = k \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta$$

where k is a constant to be determined.

(5)

(b) Hence find, by integration, the exact area of R .

(3)

$$a) \int_0^3 \sqrt{(3-x)(x+1)} dx$$

$$x = 1 + 2\sin\theta$$

$$\frac{dx}{d\theta} = 2\cos\theta \quad dx = 2\cos\theta d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4(1-\sin\theta)(1+\sin\theta)} \times 2\cos\theta d\theta$$

$$3-x = 2-2\sin\theta = 2(1-\sin\theta)$$

$$x+1 = 2+2\sin\theta = 2(1+\sin\theta)$$

$$= 4 \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta} \times \cos\theta d\theta$$

$$x=3$$

$$1+2\sin\theta = 3$$

$$2\sin\theta = 2$$

$$\sin\theta = 1$$

$$\theta = \frac{\pi}{2}$$

$$= 4 \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\cos^2\theta} \times \cos\theta d\theta$$

$$x=0$$

$$1+2\sin\theta = 0$$

$$2\sin\theta = -1$$

$$\sin\theta = -\frac{1}{2}$$

$$\theta = -\frac{\pi}{6}$$

$$= 4 \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta d\theta$$

$$b) \cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \frac{1}{2}(\cos 2\theta + 1) = \cos^2\theta$$

$$\Rightarrow 4 \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta d\theta = 2 \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos 2\theta + 1) d\theta = 2 \left[\frac{1}{2} \sin 2\theta + \theta \right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2 \times \frac{1}{2} \times [\sin 2\theta + 2\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} = [\sin 2\theta + 2\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left(0 + \pi \right) - \left(-\frac{\sqrt{3}}{2} - \frac{\pi}{3} \right) = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

7. (a) Express $\frac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P(P-2)\cos 2t, \quad t \geq 0$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(b) solve this differential equation to show that

$$P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}$$

(7)

(c) find the time taken for the population to reach 4000 for the first time.
Give your answer in years to 3 significant figures.

(3)

$$a) \quad \frac{2}{P(P-2)} = \frac{A}{P} + \frac{B}{P-2} \Rightarrow 2 = A(P-2) + BP$$

$$P=0 \Rightarrow -2A=2 \therefore A=-1 \quad P=2 \Rightarrow 2=2B \therefore B=1$$

$$= \frac{1}{P-2} - \frac{1}{P}$$

$$b) \quad \int \frac{2}{P(P-2)} dP = \int \cos 2t dt$$

$$\int \frac{1}{P-2} - \frac{1}{P} dP = \frac{1}{2} \sin 2t + C$$

$$\Rightarrow \ln(P-2) - \ln P = \ln\left(\frac{P-2}{P}\right) = \frac{1}{2} \sin 2t + C$$

$$P=3, t=0 \Rightarrow \ln\left(\frac{1}{3}\right) = \frac{1}{2} \sin 0 + c \quad \therefore c = -\ln 3$$

$$\Rightarrow \ln\left(\frac{P-2}{P}\right) + \ln 3 = \frac{1}{2} \sin 2t$$

$$\Rightarrow \frac{1}{2} \sin 2t = \ln\left(\frac{3P-6}{P}\right) \Rightarrow \frac{3P-6}{P} = e^{\frac{1}{2} \sin 2t}$$

$$\Rightarrow 3P-6 = P e^{\frac{1}{2} \sin 2t} \Rightarrow 3P - P e^{\frac{1}{2} \sin 2t} = 6$$

$$\Rightarrow P(3 - e^{\frac{1}{2} \sin 2t}) = 6 \quad \therefore P = \frac{6}{3 - e^{\frac{1}{2} \sin 2t}}$$

$$c) \frac{6}{3 - e^{\frac{1}{2} \sin 2t}} = 4000 \Rightarrow 3 - e^{\frac{1}{2} \sin 2t} = \frac{3}{2000}$$

$$\therefore e^{\frac{1}{2} \sin 2t} = 2.9985$$

$$\therefore \sin 2t = 2 \ln(2.9985)$$

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$$c) \frac{6}{3 - e^{\frac{1}{2} \sin 2t}} = 4 \Rightarrow 3 - e^{\frac{1}{2} \sin 2t} = \frac{3}{2}$$

$$\therefore e^{\frac{1}{2} \sin 2t} = \frac{3}{2} \quad \therefore \sin 2t = 2 \ln\left(\frac{3}{2}\right)$$

$$\therefore 2t = 0.9457 \sim$$

$$\therefore t = 0.473$$

2

8.

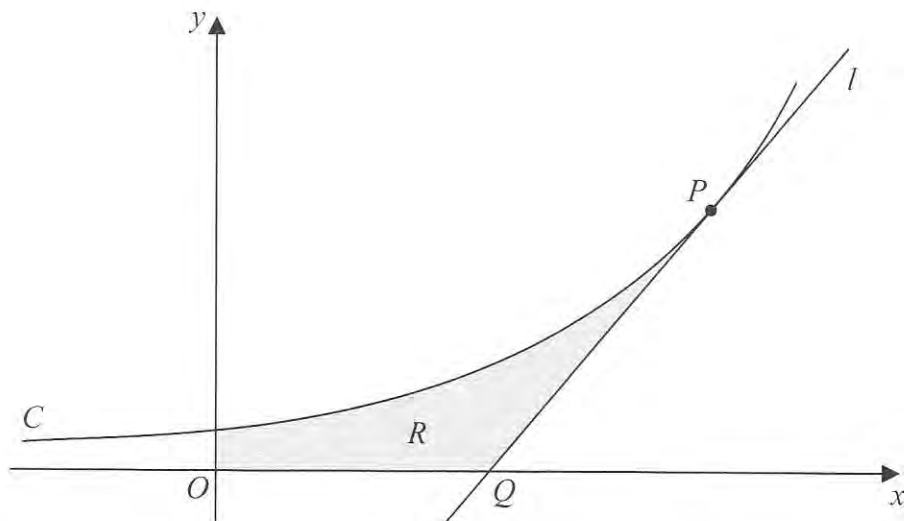
Diagram
not to scale

Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = 3^x$$

The point P lies on C and has coordinates $(2, 9)$.

The line l is a tangent to C at P . The line l cuts the x -axis at the point Q .

(a) Find the exact value of the x coordinate of Q .

(4)

The finite region R , shown shaded in Figure 3, is bounded by the curve C , the x -axis, the y -axis and the line l . This region R is rotated through 360° about the x -axis.

(b) Use integration to find the exact value of the volume of the solid generated.

Give your answer in the form $\frac{p}{q}$ where p and q are exact constants.

[You may assume the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a cone.]

(6)

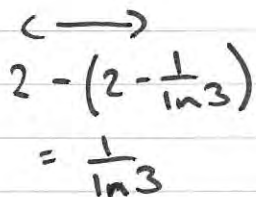
$$y = 3^x \Rightarrow \frac{dy}{dx} = 3^x \ln 3 \quad \text{when } x=2 \text{ M}_t \text{ at } P = 9 \ln 3$$

$$y - 9 = 9 \ln 3 (x - 2) \quad \text{at } Q \quad y = 0$$

$$-9 = 9 \ln 3 (x - 2) \Rightarrow x - 2 = \frac{-9}{9 \ln 3} \Rightarrow x = 2 - \frac{1}{\ln 3}$$

$$x = \frac{(2 \ln 3) - 1}{\ln 3}$$

$$b) \pi \int_0^2 y^2 dx - \text{cone } r=9$$



$$2 - \left(2 - \frac{1}{\ln 3}\right) = \frac{1}{\ln 3}$$

$$= \pi \int_0^2 (3^x)^2 dx - \frac{1}{3} \pi (9)^2 \times \frac{1}{\ln 3}$$

$$= \pi \int_0^2 3^{2x} dx - \frac{81\pi}{3 \ln 3} \int 3^{2x} dx$$

$$\text{let } u=2x \quad \frac{du}{dx} = 2$$

$$dx = \frac{1}{2} du$$

$$\frac{1}{2} \int 3^u du$$

$$= \frac{1}{2} \frac{3^u}{\ln 3} = \frac{3^{2x}}{2 \ln 3}$$

$$= \pi \left[\frac{3^{2x}}{2 \ln 3} \right]_0^2 - \frac{81\pi}{3 \ln 3}$$

$$= \pi \left[\frac{81}{2 \ln 3} - \frac{1}{2 \ln 3} \right] - \frac{81\pi}{3 \ln 3}$$

$$= \frac{80\pi}{2 \ln 3} - \frac{81\pi}{3 \ln 3}$$

$$= \frac{40\pi}{\ln 3} - \frac{27\pi}{\ln 3} = \frac{13\pi}{\ln 3}$$