June 2007 Core 4

JUNE 2007 CORE4

Q1 $f(x) = (3+2x)^{-3}$

$$(3+2x)^{-3} = 3^{-3}\left(1+\frac{2}{3}x\right)^{-3}$$
$$= \frac{1}{27}\left(1+\frac{2}{3}x\right)^{-3}$$

$$f(x) = \frac{1}{27}\left(1+(-3)\left(\frac{2}{3}x\right) + \frac{(-3)(-3-1)}{2!}\left(\frac{2}{3}x\right)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}\left(\frac{2}{3}x\right)^3\right)$$

$$= \frac{1}{27}\left(1 - 2x + \frac{8}{3}x^2 - \frac{80}{27}x^3 + \dots\right)$$

$$\frac{(\cancel{-3})(-4)}{\cancel{2}} \times \frac{2^2}{3^2} = \frac{8}{3}$$

$$\frac{(\cancel{-3})(-4)(-5)}{\cancel{2\times 3}} \times \frac{2^3}{3^3} = \frac{80}{27}$$

$$= \frac{1}{27} - \frac{2}{27}x + \frac{8}{81}x^2 - \frac{80}{729}x^3 + \dots$$

$$f(x) = \frac{1}{27} - \frac{2}{27}x + \frac{8}{81}x^2 - \frac{80}{729}x^3 + \dots$$

Q2

$u = 2^x$ $\qquad \int_0^1 \frac{2^x}{(2^x+1)^2}\,dx$

$\frac{du}{dx} = 2^x \ln 2$ $\qquad = \int_{u(0)}^{u(1)} \frac{du}{\ln 2 (u+1)^2}$ $\qquad u(1) = 2^1 = 2$, $u(0) = 2^0 = 1$

$\frac{du}{\ln 2} = 2^x dx$

$$= \frac{1}{\ln 2}\int_1^2 (u+1)^{-2}\,du$$
$$= \frac{1}{\ln 2}\left[-\frac{1}{u+1}\right]_1^2$$
$$= \frac{1}{\ln 2}\left(-\frac{1}{3} - \left(-\frac{1}{2}\right)\right) = \frac{1}{6\ln 2}$$

Q3 (a) $\int x\cos 2x\,dx$

Integration by parts: $u = x$ $\qquad \frac{dv}{dx} = \cos 2x$

$\frac{du}{dx} = 1$ $\qquad v = \frac{1}{2}\sin 2x$

$$I = x \times \frac{1}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,dx + c$$
$$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$$

(b) $\int x\cos^2 x\,dx$

$\cos 2x = 2\cos^2 x - 1$
$2\cos^2 x = \cos 2x + 1$
$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$

$$= \frac{1}{2}\int (x\cos 2x + x)\,dx$$
$$= \frac{1}{2}\left\{\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + \frac{1}{2}x^2\right\} + C$$
$$= \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + \frac{1}{4}x^2 + C$$

Q4 $\frac{2(4x^2+1)}{(2x+1)(2x-1)} \equiv A + \frac{B}{(2x+1)} + \frac{C}{(2x-1)}$

(a) $\equiv \frac{A(2x+1)(2x-1) + B(2x-1) + C(2x+1)}{(2x+1)(2x-1)}$

So $2(4x^2+1) \equiv A(2x+1)(2x-1) + B(2x-1) + C(2x+1)$

$x = -\frac{1}{2}$ $\qquad 4 = B(2\times(-\frac{1}{2}) - 1)$
$4 = -2B$
$B = -2$

$x = \frac{1}{2}$ $\qquad 4 = C(2\times\frac{1}{2}+1)$
$4 = 2C$
$C = 2$

$x = 0$ $\qquad 2 = -A - B + C$
$2 = -A + 2 + 2$ (substitute $B=-2$, $C=2$)
$A = 2$

(b) $I = \int_1^2 \left(2 - \frac{2}{(2x+1)} + \frac{2}{(2x-1)}\right)dx = \left[2x - \ln(2x+1) + \ln(2x-1)\right]_1^2$

$\int \frac{2}{2x+1}\,dx = 2\times\frac{1}{2}\ln(2x+1) = \ln(2x+1)$

$\int \frac{2}{2x-1}\,dx = 2\times\frac{1}{2}\ln(2x-1) = \ln(2x-1)$

$$= \left[2x + \ln\frac{2x-1}{2x+1}\right]_1^2$$
$$= \left[2\times 2 + \ln\frac{2\times 2-1}{2\times 2+1}\right] - \left[2 + \ln\frac{2-1}{2+1}\right]$$

$$= 4 + \ln\frac{3}{5} - 2 - \ln\frac{1}{3} = 2 + \ln\left(\frac{3}{5}\div\frac{1}{3}\right) = 2 + \ln\left(\frac{9}{5}\right) \qquad k = \frac{9}{5}$$

Q5 $l_1: \underline{r} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$ $\qquad l_2: \underline{r} = \begin{pmatrix}1\\3\\6\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$

(a) Show $l_1 \neq l_2$

Assume they meet: $l_1 = l_2$

$$\begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\6\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$\begin{pmatrix}1+\lambda\\ \lambda\\ -1\end{pmatrix} = \begin{pmatrix}1+2\mu\\ 3+\mu\\ 6-\mu\end{pmatrix}$$

$1+\lambda = 1+2\mu$ (1)
$\lambda = 3+\mu$ (2)
$-1 = 6-\mu$ (3)

Using (3) $\mu = 7$

Substitute into (1) $\lambda = 3+7 \Rightarrow \lambda = 10$
(2) $1+\lambda = 1+2\times 7 \Rightarrow \lambda = 14$

contradiction

Hence they never meet.

(b) A: $\lambda = 1$ $\qquad \underline{a} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$

B: $\mu = 2$ $\qquad \underline{b} = \begin{pmatrix}1\\3\\6\end{pmatrix} + 2\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\3\\6\end{pmatrix} + \begin{pmatrix}4\\2\\-2\end{pmatrix} = \begin{pmatrix}5\\5\\4\end{pmatrix}$

$$\vec{AB} = \underline{b} - \underline{a} = \begin{pmatrix}5\\5\\4\end{pmatrix} - \begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}3\\4\\5\end{pmatrix}$$

direction of l_1: $\underline{d}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$

Q5 (b) Using the dot product between AB and d_1:

$$\cos\alpha = \frac{AB\cdot d_1}{|AB|\times|d_1|}$$

$$= \frac{\begin{pmatrix}3\\4\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\left|\begin{pmatrix}3\\4\\5\end{pmatrix}\right|\times\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\right|}$$

$$= \frac{3+4}{\sqrt{(3^2+4^2+5^2)}\times\sqrt{(1^2+1^2+0^2)}}$$

$$= \frac{7}{\sqrt{50}\times\sqrt{2}} = \underline{\frac{7}{10}}$$

Q6

$x = \tan^2 t \quad , \quad y = \sin t \qquad 0 < t < \frac{\pi}{2}$

(a) $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$\frac{dx}{dt} = \cancel{2\tan t}$ $(\tan x)' = \sec^2$

$= 2\tan t \times \sec^2 t$

$\frac{dy}{dt} = \cos t$

$= \cos t \div (2\tan t\sec^2 t)$

$= \frac{\cos t}{2\tan t\sec^2 t}$

$= \frac{\cos t}{\frac{2\sin t}{\cos t}\times\frac{1}{\cos^2 t}} = \frac{\cos^4 t}{2\sin t}$

(b) need gradient:

$m = \frac{dy}{dx}\Big|_{t=\frac{\pi}{4}} = \frac{\cos t}{2\tan t\sec^2 t}\Big|_{t=\frac{\pi}{4}} = \frac{\cos\frac{\pi}{4}}{2\tan\frac{\pi}{4}\sec^2\frac{\pi}{4}}$

$= \frac{\frac{\sqrt{2}}{2}}{2\times1\times2} = \frac{\sqrt{2}}{8}$

need the point:

$x_1\big|_{t=\frac{\pi}{4}} = \tan^2\frac{\pi}{4} = 1 \qquad y_1\big|_{t=\frac{\pi}{4}} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$y - y_1 = m(x - x_1)$

$y - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{8}(x-1)$

$y = \frac{\sqrt{2}}{8}x - \frac{\sqrt{2}}{8} + \frac{\sqrt{2}^{\times4}}{2_{\times4}}$

$\underline{\underline{y = \frac{\sqrt{2}}{8}x + \frac{3\sqrt{2}}{8}}}$

Q6 (c) $x = \tan^2 t \qquad y^2 = \sin^2 t$

$x = \frac{\sin^2 t}{\cos^2 t} \qquad \sin^2 t + \cos^2 t \equiv 1$

$x = \frac{\sin^2 t}{1-\sin^2 t}$

$x = \frac{y^2}{(1-y^2)}$

$x(1-y^2) = y^2$

$x - xy^2 = y^2$

$y^2 + xy^2 = x$

$y^2(1+x) = x$

$\underline{y^2 = \frac{x}{1+x}} \qquad (4)$

Q7 $y = \sqrt{\tan x}$

(a)

x	0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$
y	0	0.4459959	0.64359	0.81742	1

(b) $I = \int_0^{\frac{\pi}{4}}\sqrt{\tan x}\,dx \approx \frac{h}{2}\left[y_0+y_4+2(y_1+y_2+y_3)\right]$

$= \frac{\frac{\pi}{16}}{2}\left[0+1+2(0.445995+0.64359+0.81742)\right]$

$= 0.4726$

Hence the area of R: $\underline{\underline{A_R = 0.4726}}$

(c) Volume $= \pi\int_0^{\frac{\pi}{4}} y^2\,dx$

$= \pi\int_0^{\frac{\pi}{4}}\tan x\,dx$

$= \pi\left[\ln\sec x\right]_0^{\frac{\pi}{4}}$

$\ln\sec\frac{\pi}{4} = \ln\sqrt{2} = \frac{1}{2}\ln 2$

$\ln 1 = 0$

$= \pi\left[\ln\sec\frac{\pi}{4} - \ln\sec 0\right]$

$= \pi\left(\frac{1}{2}\ln 2\right)$

$= \underline{\frac{\pi}{2}\ln 2}$

Q8 $\frac{dP}{dt} = kP$ at $t=0$ $P=P_0$

(a)

Separating the variables:

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln P = kt + C$$

$t=0, P=P_0 \Rightarrow \ln P_0 = C$

So $\ln P = \ln P_0 + kt$

$$\ln P - \ln P_0 = kt$$

$$\ln \frac{P}{P_0} = kt$$

$$\frac{P}{P_0} = e^{kt}$$

$$\underline{\underline{P = P_0 e^{kt}}} \quad (1)$$

(b) $P = 2P_0$ and $k = 2.5$

so $2P_0 = P_0 e^{2.5t}$ (substituting into (1))

$$e^{2.5t} = 2$$

$$2.5t = \ln 2$$

$$\underline{\underline{t = 0.4\ln 2}} \quad [= 0.2772588 \text{ days} = 399 \text{ min}]$$

Q8 $\frac{dP}{dt} = \lambda P\cos\lambda t$

separating variables: $\int \frac{dP}{P} = \lambda \int \cos\lambda t\,dt$

$$\ln P = \frac{\lambda}{\lambda}\sin\lambda t + C$$

$$\ln P = \sin\lambda t + c$$

$$P = e^{\sin\lambda t + c} \quad (e^c \times e^{\sin\lambda t})$$

$$P = Ae^{\sin\lambda t}$$

at $t=0$ $P=P_0$: $P_0 = Ae^{(\sin 0)}$

$$P_0 = A$$

So $\underline{\underline{P = P_0 e^{\sin\lambda t}}}$ (4)

(d) $P = 2P_0 \Rightarrow 2\cancel{P_0} = \cancel{P_0} e^{\sin\lambda t}$

$$\sin\lambda t = \ln 2$$

$$\lambda t = \arcsin(\ln 2)$$

$$t = \frac{1}{\lambda}\arcsin(\ln 2)$$

$\lambda = 2.5 \Rightarrow t = 0.4\arcsin(\ln 2)$

$= 0.3063384$ days ($\times 24 \times 60$

$\underline{\underline{= 441 \text{ minutes}}}$