CORE 4 JUNE 05

(Q1) $(4-9x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1-\frac{9x}{4}\right)^{\frac{1}{2}} = 2\left(1-\frac{9x}{4}\right)^{\frac{1}{2}}$

$= 2\left(1 + \frac{1}{2}\left(-\frac{9x}{4}\right) + \frac{\frac{1}{2}\times\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{9x}{4}\right)^2 + \frac{\frac{1}{2}\times\left(\frac{1}{2}-1\right)\times\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{9x}{4}\right)^3\right)$

$= 2\left(1 - \frac{9}{8}x + \frac{\frac{1}{2}\times\left(-\frac{1}{2}\right)}{2}\times\frac{81x^2}{16} + \frac{\frac{1}{2}\times\left(-\frac{1}{2}\right)\times\left(-\frac{3}{2}\right)}{2\times 3}\times\frac{-729x^3}{64}\right)$

$= 2\left(1 - \frac{9}{8}x - \frac{81}{128}x^2 - \frac{729}{1024}x^3\right)$

$= 2 - \frac{9}{4}x - \frac{81}{64}x^2 - \frac{729}{512}x^3 + \ldots$

(Q2) Differentiating $x^2+2xy-3y^2+16=0$ (i) implicitly with respect to x:

$2x + \left(2y + 2x\frac{dy}{dx}\right) - 6y\frac{dy}{dx} = 0$

$2x + 2y + (2x-6y)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2x+2y}{(2x-6y)} = \frac{x+y}{3y-x}$

$\frac{dy}{dx} = 0 \Rightarrow x+y=0$ the numerator $=0$

$y = -x$ (ii)

Substitute $y=-x$ into equation (i)

$x^2 - 2x^2 - 3x^2 + 16 = 0$

(Q2) ctd

$4x^2 = 16$

$x^2 = 4$

$x = \pm 2$

$y = \mp 2$

$(2,-2)$ or $(-2,2)$

(Q3) a) Let $\frac{5x+3}{(2x-3)(x+2)} \equiv \frac{A}{(2x-3)} + \frac{B}{(x+2)}$

$\equiv \frac{A(x+2) + B(2x-3)}{(2x-3)(x+2)}$

COMPARING THE NUMERATORS:

$5x+3 \equiv A(x+2) + B(2x-3)$

Let $x=-2 \Rightarrow 5\times -2 + 3 = B(2\times -2 - 3)$

$-7 = -7B$

$B = +1$

Achieved by making $(x+2)=0$ and $(2x-3)=0$

Let $x = \frac{3}{2} \Rightarrow 5\times\frac{3}{2} + 3 = A\left(\frac{3}{2}+2\right)$

$\frac{21}{2} = \frac{7}{2}A$

$A = 3$

(Q3) (b)

$\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,dx$

$= \int_2^6 \left(\frac{3}{2x-3} + \frac{1}{x+2}\right)dx$

$= \left[\frac{3}{2}\ln|2x-3| + \ln|x+2|\right]_2^6$

$= \left[\frac{3}{2}\ln(2\times6-3) + \ln(6+2)\right] - \left[\frac{3}{2}\ln(2\times2-3) + \ln(2+2)\right]$

$= \frac{3}{2}\ln 9 + \ln 8 - \frac{3}{2}\ln 1 - \ln 4$

$= \ln 9^{\frac{3}{2}} + \ln 8 - 0 - \ln 4$

$= \ln\left(\frac{27\times 8}{4}\right)$

$= \ln 54$

(Q4) $x = \sin\theta$ $\quad \frac{dx}{d\theta} = \cos\theta$

$I = \int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$

$dx = \cos\theta\,d\theta$

$x^2 = \sin^2\theta$

$1-x^2 = 1-\sin^2\theta$

$1-x^2 = \cos^2\theta$

$(1-x^2)^{\frac{3}{2}} = \cos^3\theta$

(Q4)

$I = \int_{\theta(0)}^{\theta(\frac{1}{2})} \frac{1}{\cos^3\theta}\cos\theta\,d\theta$

Change the limit:

$x = \sin\theta$

$\theta\left(\frac{1}{2}\right) \Rightarrow \frac{1}{2} = \sin\theta$

$\theta = \frac{\pi}{6}$

$\theta(0) \Rightarrow 0 = \sin\theta$

$\theta = 0$

for θ acute;

$= \int_0^{\frac{\pi}{6}} \frac{d\theta}{\cos^2\theta}$

$= \int_0^{\frac{\pi}{6}} \sec^2\theta\,d\theta$

$= \left[\tan\theta\right]_0^{\frac{\pi}{6}} = \left[\tan\frac{\pi}{6} - \tan 0\right]$

$= \frac{\sqrt{3}}{3}$

(Q5) (a) Area $= \int_0^1 xe^{2x}\,dx$

INTEGRATION BY PARTS:

$u = x \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1 \quad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}xe^{2x} - \int\frac{1}{2}e^{2x}\,dx$

$= \left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1$

$= \left[\frac{1}{2}\times1\times e^2 - \frac{1}{4}e^2\right] - \left[\frac{1}{2}\times0\times e^0 - \frac{1}{4}e^0\right]$

$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}(e^2+1)$

Q5 (b) $y = xe^{2x}$

x	0	0.2	0.4	0.6	0.8	1
y	0	0.29836	0.890216	1.99207	3.96243	7.38906

(c) $\text{Area} = \int_0^1 y\,dx \approx \frac{h}{2}\left[y_0 + y_n + 2(y_1 + \dots + y_{n-1})\right]$

with $h = 0.2$

$$= \frac{0.2}{2}\left[0 + 7.38906 + 2(0.29836 + 0.890216 + 1.99207 + 3.96243)\right]$$

$$= 2.1675212$$

$$\underline{= 2.168 \quad (4sf)}$$

Q6

$x = 2\cot t \qquad y = 2\sin^2 t$

(a) $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ $\qquad \frac{dx}{dt} = -2\operatorname{cosec}^2 t$

$\qquad \frac{dy}{dt} = 4\sin t\cos t$

$$= \frac{\cancel{4}^{2}\sin t\cos t}{-\cancel{2}\operatorname{cosec}^2 t}$$

$$= \frac{-2\sin t\cos t}{\frac{1}{\sin^2 t}}$$

$$\underline{\underline{= -2\sin^3 t\cos t}}$$

Q6 (b) m_t – gradient of the tangent:

$$m_t = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = -2\sin^3\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) = -2\times\left(\frac{\sqrt{2}}{2}\right)^3\times\frac{\sqrt{2}}{2} = -2\times\frac{1}{4} = -\frac{1}{2}$$

Coordinates of a point when $t = \frac{\pi}{4}$:

$x\left(\frac{\pi}{4}\right) = 2\cot\frac{\pi}{4} = 2 \qquad y\left(\frac{\pi}{4}\right) = 2\sin^2\frac{\pi}{4} = 2\times\frac{1}{2} = 1$

$y - y_1 = m(x - x_1)$

$y - 1 = -\frac{1}{2}(x - 2)$

$$\underline{y = -\frac{1}{2}x + 2}$$

(c) $x = 2\cot t \qquad y = 2\sin^2 t$

$x^2 = 4\frac{\cos^2 t}{\sin^2 t} \qquad \frac{y}{2} = \sin^2 t$

$x^2\sin^2 t = 4\cos^2 t$

$x^2\sin^2 t = 4(1 - \sin^2 t)$

$x^2\times\frac{y}{2} = 4\left(1 - \frac{y}{2}\right) \quad /\times 2$

$yx^2 = 8 - 4y$

$y(x^2 + 4) = 8$

$$\underline{y = \frac{8}{x^2+4}}$$

Q7 $l_1: \underline{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\4\end{pmatrix} \qquad l_2: \underline{r} = \begin{pmatrix}0\\4\\-2\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\0\end{pmatrix}$

(a) $l_1 \cdot l_2$: $\begin{pmatrix}3+\lambda\\1-\lambda\\2+4\lambda\end{pmatrix} = \begin{pmatrix}\mu\\4-\mu\\-2\end{pmatrix}$ $\begin{array}{l}\text{i}\\\text{ii}\\\text{iii}\end{array}$ $\quad B(2, 2, -2)$

from (iii) $\quad 2 + 4\lambda = -2 \Rightarrow \underline{\lambda = -1}$

Into (i) $\quad 3 + (-1) = \mu \Rightarrow \underline{\mu = 2}$

Checking with (ii) $\quad 1 - (-1) = 4 - 2 \Rightarrow 2 = 2$ ✓

(b) θ is the angle between the two direction vectors of the lines:

of l_1: $d_1 = \begin{pmatrix}1\\-1\\4\end{pmatrix}$ and of l_2: $d_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

$$\cos\theta = \frac{d_1\cdot d_2}{|d_1|\times|d_2|} = \frac{\begin{pmatrix}1\\-1\\4\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix}}{\sqrt{1^2+(-1)^2+4^2}\times\sqrt{1^2+(-1)^2}}$$

$$= \frac{1+1}{\sqrt{18}\times\sqrt{2}} = \frac{2}{6} = \underline{\underline{\frac{1}{3}}}$$

(c) $\underline{a} = 3i + j + 2k \qquad \underline{c} = 5i - j - 2k$

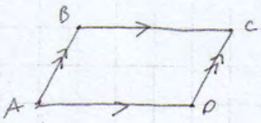
$\underline{b} = 2i + 2j - 2k$

$\overrightarrow{AB} = \underline{b} - \underline{a} = 2i + 2j - 2k - 3i - j - 2k = -i + j - 4k$

$\overrightarrow{BC} = \underline{c} - \underline{b} = 5i - j - 2k - 2i - 2j + 2k = 3i - 3j$

$|\overrightarrow{AB}| = \sqrt{(-1)^2 + 1^2 + (-4)^2} = \sqrt{18} \qquad |\overrightarrow{BC}| = \sqrt{3^2 + (-3)^2} = \sqrt{18}$

(d) from (c) $|\overrightarrow{AB}| = |\overrightarrow{BC}|$



Since ABCD is a parallelogram $\overrightarrow{AB} = \overrightarrow{DC}$

$\overrightarrow{AB} = -i + j - 4k \qquad$ Let $D(x, y, z)$

$\overrightarrow{DC} = (5 - x)i + (-1 - y)j + (-2 - z)k$

Comparing the components:

is : $-1 = 5 - x \Rightarrow x = 6$

js : $1 = -1 - y \Rightarrow y = -2$

ks : $-4 = -2 - z \Rightarrow z = 2$

$$\overrightarrow{OD} = \underline{d} = \underline{6i - 2j + 2z}$$

Q8 (a) $\frac{dV}{dt}$ ↖ the rate of change of the volume

Pouring in with a constant rate $20\,cm^3 s^{-1}$
Pouring out (negative sign indicating the loss): $\propto V$
$-kV$ (directly proportional to V)

$$\frac{dV}{dt} = 20 - kV$$

(b) At $t=0$ $V=0$

$$\frac{dV}{dt} = 20 - kV$$

Separating the variables:

$$\int \frac{dV}{20-kV} = \int dt$$

$$-\frac{1}{k}\ln|20-kV| = t + C$$

$V=0$, $t=0$ $\Rightarrow$ $-\frac{1}{k}\ln 20 = C$

$$-\frac{1}{k}\ln(20-kV) = t - \frac{1}{k}\ln 20$$

$$\frac{1}{k}\ln(20-kV) - \frac{1}{k}\ln 20 = -t \qquad / \times k$$

$$\ln\left(\frac{20-kV}{20}\right) = -kt$$

Q8(b)

$$\frac{20-kV}{20} = e^{-kt}$$

$$20 - kV = 20e^{-kt}$$

$$kV = 20 - 20e^{-kt}$$

$$V = \frac{1}{k}\left(20 - 20e^{-kt}\right) \quad (i)$$

$$V = \frac{20}{k} - \frac{20}{k}e^{-kt}$$

$$A = \frac{20}{k} \qquad B = -\frac{20}{k}$$

(c) $\frac{dV}{dt} = 10$ when $t=5$

$\frac{dV}{dt} = 20 - kV$ substituting V from (i)

$$\frac{dV}{dt} = 20 - k \times \frac{1}{k}\left(20 - 20e^{-kt}\right)$$

$$\frac{dV}{dt} = 20 - 20 + 20e^{-kt}$$

$$\frac{dV}{dt} = 20e^{-kt}$$

So

$$10 = 20e^{-5k}$$

$$0.5 = e^{-5k}$$

$$\ln 0.5 = -5k \qquad (\ln 0.5 = \ln(2^{-1}) = -\ln 2$$

$$-\ln 2 = -5k$$

$$k = \frac{1}{5}\ln 2$$

At $t=10$ $V = \frac{1}{\frac{1}{5}\ln 2}\left(20 - 20e^{\left(-\frac{1}{5}\ln 2\right)\times 10}\right) =$

At $t=10$

$$V = \frac{1}{\frac{1}{5}\ln 2}\left(20 - 20e^{\left(-\frac{1}{5}\ln 2\right)\times 10}\right)$$

$$= \frac{5}{\ln 2}\left(20 - 20e^{-2\ln 2}\right)$$

$$= \frac{5}{\ln 2}\left(20 - 20e^{\ln 2^{-2}}\right)$$

$$= \frac{5}{\ln 2}\left(20 - 20 \times \frac{1}{4}\right)$$

$$= \frac{75}{\ln 2}$$