C4 JAN 10

1. (a) Find the binomial expansion of

$$\sqrt{(1-8x)}, \qquad |x| < \frac{1}{8},$$

in ascending powers of x up to and including the term in x^3, simplifying each term.

(4)

(b) Show that, when $x = \frac{1}{100}$, the exact value of $\sqrt{(1-8x)}$ is $\frac{\sqrt{23}}{5}$.

(2)

(c) Substitute $x = \frac{1}{100}$ into the binomial expansion in part (a) and hence obtain an approximation to $\sqrt{23}$. Give your answer to 5 decimal places.

(3)

$$(1-8x)^{\frac{1}{2}} \simeq 1 + \tfrac{1}{2}(-8x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-8x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}(-8x)^3$$

$$\simeq 1 - 4x - 8x^2 - 32x^3$$

(b) $\sqrt{1 - \frac{8}{100}} = \sqrt{\frac{92}{100}} = \frac{\sqrt{4}\sqrt{23}}{\sqrt{100}} = \frac{2}{10}\sqrt{23} = \frac{\sqrt{23}}{5}$

(c) $\frac{\sqrt{23}}{5} \simeq 1 - \frac{4}{100} - \frac{8}{10000} - \frac{32}{1000000} = 0.959168$

$\times 5 \Rightarrow \sqrt{23} \simeq \underline{4.79584}$

2.

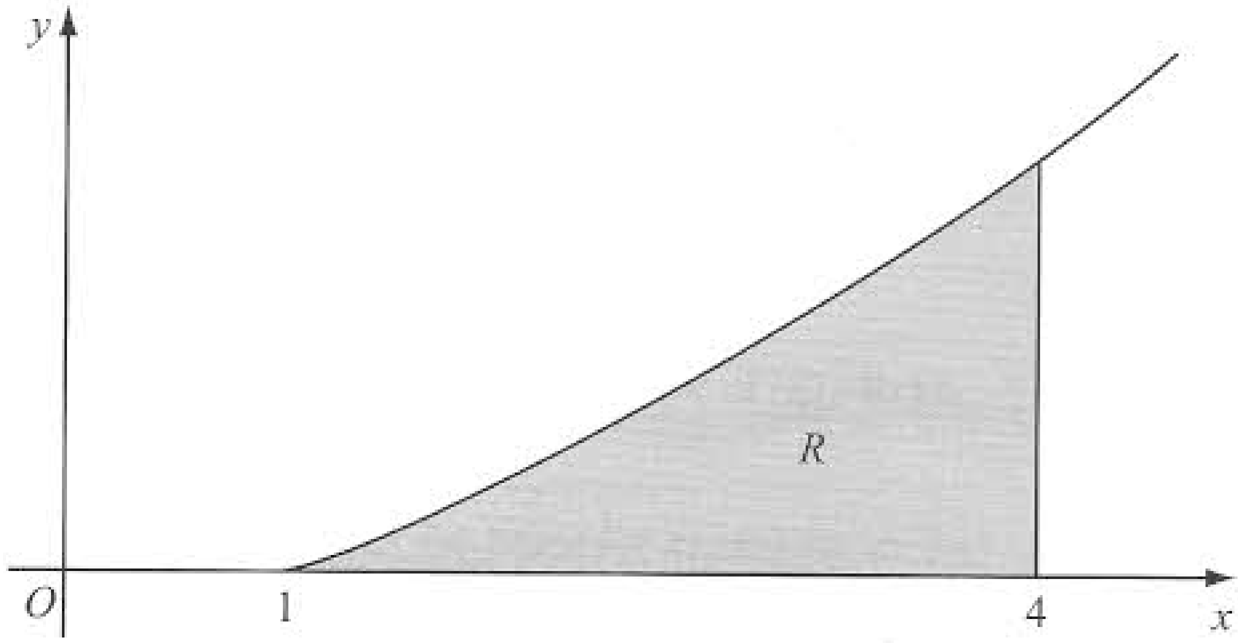


Figure 1

Figure 1 shows a sketch of the curve with equation $y = x \ln x,\ x \geqslant 1$. The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the line $x = 4$.

The table shows corresponding values of x and y for $y = x \ln x$.

x	1	1.5	2	2.5	3	3.5	4
y	0	0.608	1.386	2.291	3.296	4.385	5.545

(a) Complete the table with the values of y corresponding to $x = 2$ and $x = 2.5$, giving your answers to 3 decimal places.

(2)

(b) Use the trapezium rule, with all the values of y in the completed table, to obtain an estimate for the area of R, giving your answer to 2 decimal places.

(4)

(c) (i) Use integration by parts to find $\int x \ln x \, dx$.

(ii) Hence find the exact area of R, giving your answer in the form $\frac{1}{4}(a \ln 2 + b)$, where a and b are integers.

(7)

(b) $\frac{1}{2}\left(\frac{1}{2}\right)\left(0 + 5.545 + 2\left(0.608 + \dots + 4.385\right)\right) \simeq \underline{7.37}$

(c) $u = \ln x \quad \frac{dv}{dx} = x \quad \Rightarrow \int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^{\cancel{2}} \times \frac{1}{\cancel{x}} \, dx$

$u' = \frac{1}{x} \quad v = \frac{1}{2}x^2$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$

(ii) $\left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right]_1^4 = 8 \ln 4 - 4 - \left(-\frac{1}{4}\right) = 8 \ln 2^2 - \frac{15}{4} = \frac{1}{4}(64 \ln 2 - 15)$

3. The curve C has the equation

$$\cos 2x + \cos 3y = 1, \quad -\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{4}, \quad 0 \leqslant y \leqslant \frac{\pi}{6}$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

(3)

The point P lies on C where $x = \frac{\pi}{6}$.

(b) Find the value of y at P.

(3)

(c) Find the equation of the tangent to C at P, giving your answer in the form $ax + by + c\pi = 0$, where a, b and c are integers.

(3)

(a) $\frac{d}{dx}\cos 2x + \frac{d}{dx}\cos 3y = \frac{d}{dx}(1)$

$\Rightarrow \quad -2\sin 2x - 3\sin 3y \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{2\sin 2x}{3\sin 3y}$

(b) $\cos\left(\frac{\pi}{3}\right) + \cos 3y = 1 \Rightarrow \cos 3y = 1 - \frac{1}{2} \Rightarrow 3y = \frac{\pi}{3}$ $\underline{y = \frac{\pi}{9}}$

(c) $M_t = \frac{dy}{dx}_{\left(x=\frac{\pi}{6},\, y=\frac{\pi}{9}\right)}$ $\quad -\frac{2\sin\cancel{\left(\frac{\pi}{3}\right)}}{3\sin\cancel{\left(\frac{\pi}{3}\right)}} \Rightarrow M_t = -\frac{2}{3}$

$y - \frac{\pi}{9} = -\frac{2}{3}\left(x - \frac{\pi}{6}\right) \Rightarrow 9y - \pi = -6x + \pi$

$\underline{6x + 9y - 2\pi = 0}$

4. The line l_1 has vector equation

$$\mathbf{r} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$$

and the line l_2 has vector equation

$$\mathbf{r} = \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$$

where λ and μ are parameters.

The lines l_1 and l_2 intersect at the point A and the acute angle between l_1 and l_2 is θ.

(a) Write down the coordinates of A.

(1)

(b) Find the value of $\cos\theta$.

(3)

The point X lies on l_1 where $\lambda = 4$.

(c) Find the coordinates of X.

(1)

(d) Find the vector $\overrightarrow{AX}$.

(2)

(e) Hence, or otherwise, show that $|\overrightarrow{AX}| = 4\sqrt{26}$.

(2)

The point Y lies on l_2. Given that the vector $\overrightarrow{YX}$ is perpendicular to l_1,

(f) find the length of AY, giving your answer to 3 significant figures.

(3)

(a) $l_1 = \begin{pmatrix} -6+4\lambda \\ 4-\lambda \\ -1+3\lambda \end{pmatrix}$ $\quad l_2 = \begin{pmatrix} -6+3\mu \\ 4-4\mu \\ -1+\mu \end{pmatrix}$

$\Rightarrow \cancel{-6}+4\lambda = \cancel{-6}+3\mu \quad \Rightarrow \lambda = \frac{3}{4}\mu.$

$\cancel{4}-\lambda = \cancel{4}+4\mu \qquad \lambda = -4\mu$

$\Big\}$ only possible if $\lambda = \mu = 0$

Intersect at $(-6, 4, -1)$

(b) $\cos\theta = \dfrac{a.b}{|a||b|}$ $\quad a = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ $\quad b = \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$

$a.b = 12 + 4 + 3 = 19$ $\quad |a| = \sqrt{4^2 + 1^2 + 3^2} = \sqrt{26}$

$|b| = \sqrt{3^2 + 4^2 + 1^2} = \sqrt{26}$

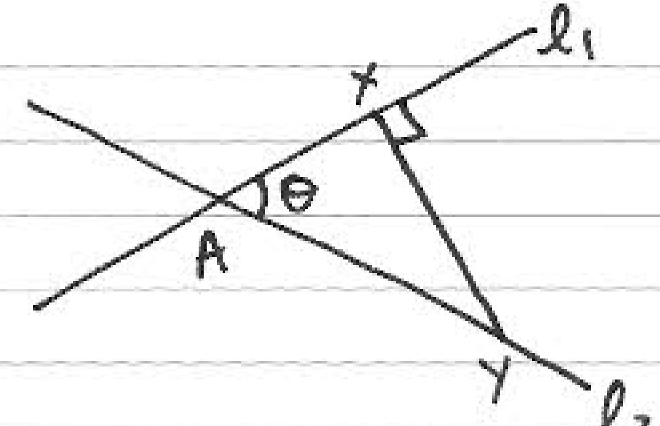
$\Rightarrow \cos\theta = \dfrac{19}{26}$

(c) $\begin{pmatrix} -6+16 \\ 4-4 \\ -1+12 \end{pmatrix} \Rightarrow (10, 0, 11)$

(a) $\overrightarrow{AX}$ $\quad \begin{pmatrix} 10 \\ 0 \\ 11 \end{pmatrix} - \begin{pmatrix} -6 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} +16 \\ -4 \\ +12 \end{pmatrix}$

(e) $|\overrightarrow{AX}| = \sqrt{16^2 + 4^2 + 12^2} = \sqrt{416} = \sqrt{16}\sqrt{26} = 4\sqrt{26}$

(f)



$\cos\theta = \dfrac{AX}{AY} \Rightarrow \dfrac{19}{26} = \dfrac{4\sqrt{26}}{AY}$

$AY = \dfrac{26 \times 4\sqrt{26}}{19} = \underline{27.9}$

5. (a) Find $\int \frac{9x+6}{x}\,\mathrm{d}x$, $x>0$. **(2)**

(b) Given that $y=8$ at $x=1$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(9x+6)\,y^{\frac{1}{3}}}{x}$$

giving your answer in the form $y^2=\mathrm{g}(x)$. **(6)**

(a) $\int 9+\frac{6}{x}\,dx = 9x+6\ln x+c$

(b) $\int y^{-\frac{1}{3}}\,dy = \int 9+\frac{6}{x}\,dx = 9x+6\ln x+c$

$\frac{3}{2}y^{\frac{2}{3}} = 9x+6\ln x+c$

$\frac{3}{2}(8)^{\frac{2}{3}} = 9+C \Rightarrow 6=9+C \quad C=-3$

$\frac{3}{2}y^{\frac{2}{3}} = 9x+6\ln x-3$

$(y^2)^{\frac{1}{3}} = 6x+4\ln x-2$

$y^2 = (6x+4\ln x-2)^3$

6. The area A of a circle is increasing at a constant rate of $1.5\,\text{cm}^2\,\text{s}^{-1}$. Find, to 3 significant figures, the rate at which the radius r of the circle is increasing when the area of the circle is $2\,\text{cm}^2$.

(5)

$$\frac{dA}{dt} = 1.5 \qquad A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r \Rightarrow \frac{dr}{dA} = \frac{1}{2\pi r}$$

$$\frac{dr}{dt} = \frac{dA}{dt}\frac{dr}{dA} = \frac{1.5}{2\pi r} = \frac{3}{4\pi r}$$

$$A = 2 = \pi r^2 \Rightarrow r = \sqrt{\frac{2}{\pi}}$$

$$\frac{dr}{dt} = \frac{3}{4\pi\sqrt{\frac{2}{\pi}}} = \underline{0.299\,\text{cms}^{-1}}$$

7.

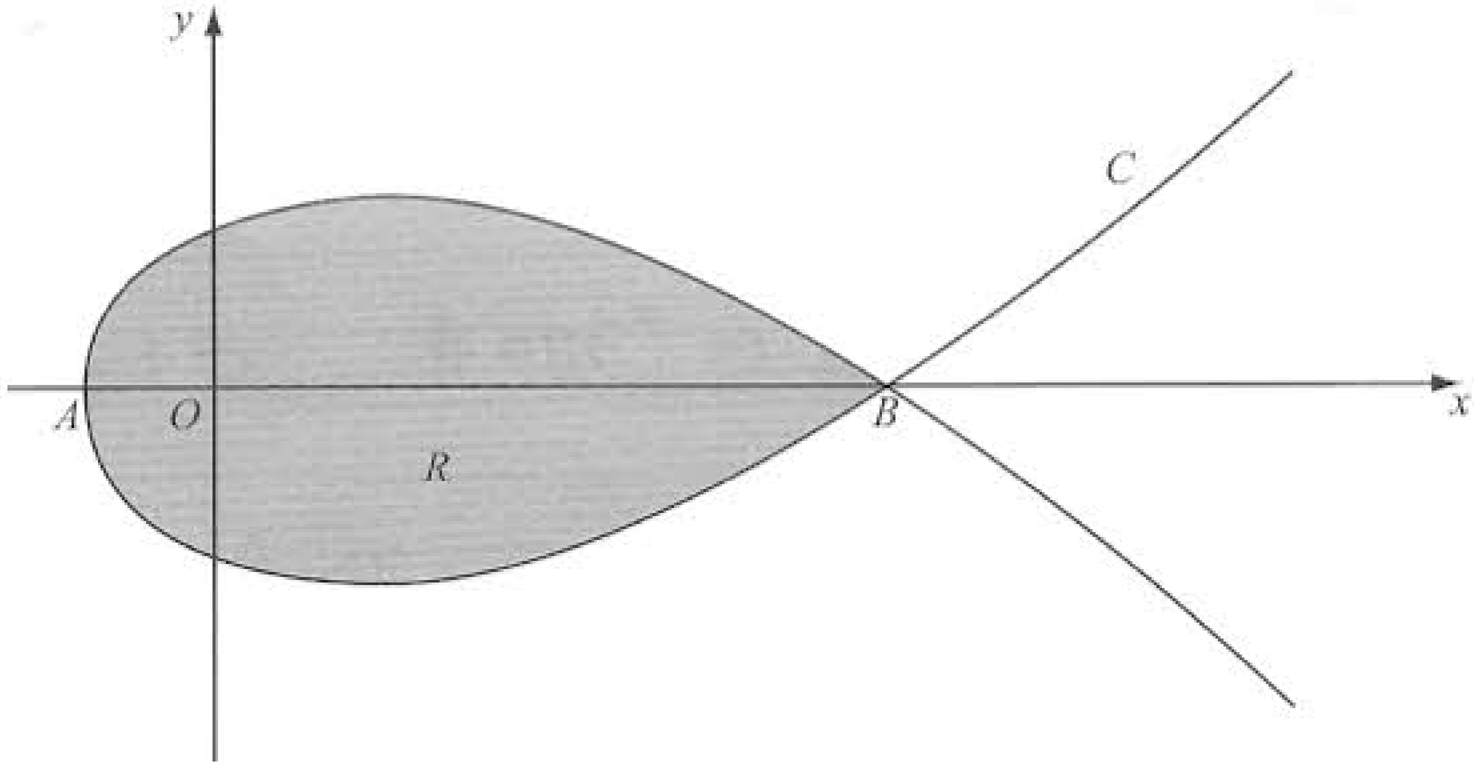


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 5t^2 - 4, \quad y = t(9 - t^2)$$

The curve C cuts the x-axis at the points A and B.

(a) Find the x-coordinate at the point A and the x-coordinate at the point B.

(3)

The region R, as shown shaded in Figure 2, is enclosed by the loop of the curve.

(b) Use integration to find the area of R.

(6)

(a) $y=0 \Rightarrow t=0 \quad t^2=9 \Rightarrow x=-4, 41$

$A(-4,0) \quad B(41,0)$

(b) $R = 2\int_{-4}^{41} y\,dx = 2\int_0^3 y\frac{dx}{dt}\,dt$

$= 2\int_0^3 t(9-t^2)\times 10t\,dt = 2\int_0^3 90t^2 - 10t^4\,dt$

$= 2\left[30t^3 - 2t^5\right]_0^3 = 2(324-0) = \underline{648}$

8. (a) Using the substitution $x = 2\cos u$, or otherwise, find the exact value of

$$\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{(4-x^2)}}\,dx$$

(7)

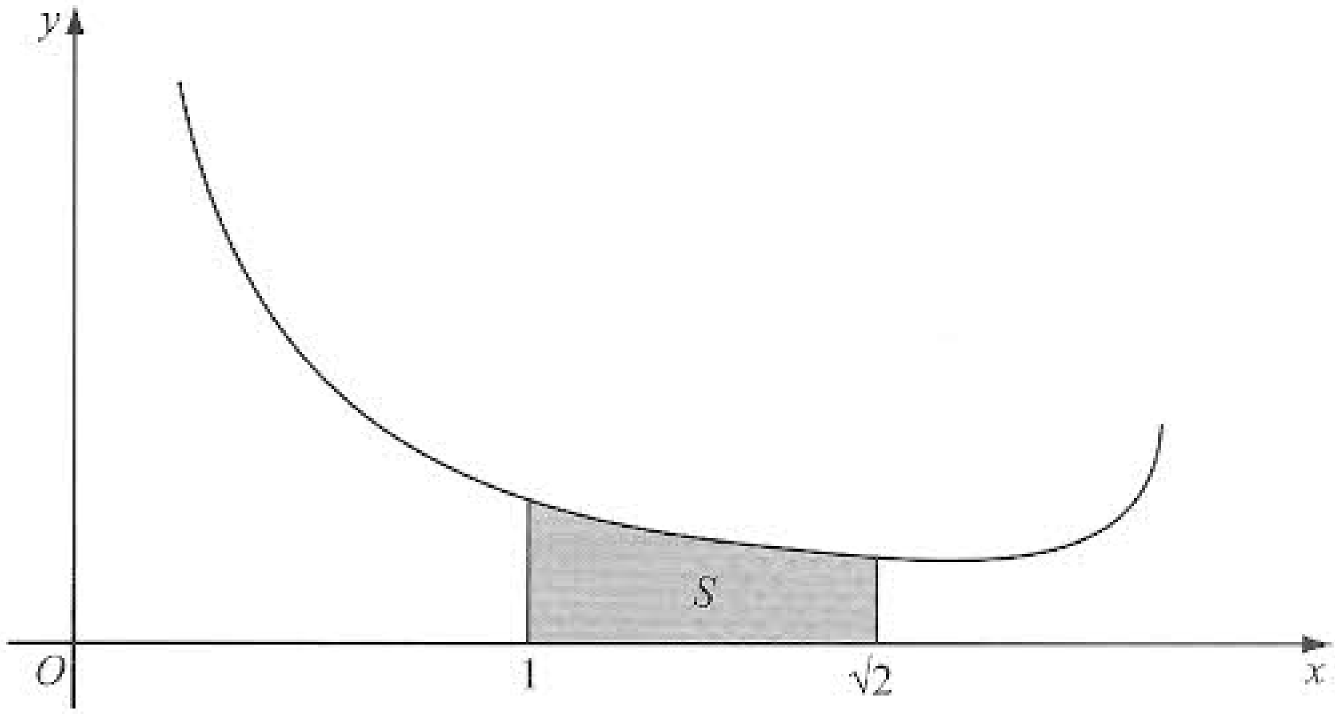


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = \dfrac{4}{x(4-x^2)^{\frac{1}{4}}}$, $\quad 0 < x < 2$.

The shaded region S, shown in Figure 3, is bounded by the curve, the x-axis and the lines with equations $x = 1$ and $x = \sqrt{2}$. The shaded region S is rotated through 2π radians about the x-axis to form a solid of revolution.

(b) Using your answer to part (a), find the exact volume of the solid of revolution formed.

(3)

(a) $x = 2\cos u$ $\qquad x^2 = 4\cos^2 u$

$\frac{dx}{du} = -2\sin u$ $\qquad 4 - x^2 = 4 - 4\cos^2 u$

$\qquad = 4(1-\cos^2 u)$

$dx = -2\sin u\,du$ $\qquad = 4\sin^2 u$

$\sqrt{4-x^2} = 2\sin u$

when $x = \sqrt{2}$ $\quad \cos u = \frac{\sqrt{2}}{2} \Rightarrow u = \frac{\pi}{4}$

$x = 1$ $\quad \cos u = \frac{1}{2} \Rightarrow u = \frac{\pi}{3}$

$$\Rightarrow \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{4\cos^2 u \times \cancel{2\sin u}} \times -\cancel{2\sin u}\, du$$

$$= -\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{\cos^2 u}\, du = -\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \sec^2 u\, du$$

$$= -\frac{1}{4}\left[\tan u\right]_{\frac{\pi}{3}}^{\frac{\pi}{4}} = -\frac{1}{4}\left(1-\sqrt{3}\right) = \frac{\sqrt{3}-1}{4}$$

(b) $$\text{Vol} = \pi\int_1^{\sqrt{2}} y^2\,dx = 2\pi\int_1^{\sqrt{2}} \frac{16}{x^2\sqrt{4-x^2}}\,dx$$

$$\text{Vol} = 16\pi \times \int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,dx$$

$$\text{Vol} = 16\pi \times \left(\frac{\sqrt{3}-1}{4}\right) = \underline{4\pi\left(\sqrt{3}-1\right)}$$