C3 JUNE 11

1. Differentiate with respect to x

(a) $\ln\left(x^2+3x+5\right)$

(2)

(b) $\dfrac{\cos x}{x^2}$

(3)

a) $\dfrac{2x+3}{x^2+3x+5}$

b) $u = \cos x \quad v = x^2 \qquad \dfrac{vu' - uv'}{v^2}$

$u' = -\sin x \quad v' = 2x$

$$\frac{dy}{dx} = \frac{x^2(-\sin x) - (\cos x)2x}{x^4}$$

$$= \frac{-x^2 \sin x - 2x\cos x}{x^4}$$ (not required)

$$= \frac{-x\sin x - 2\cos x}{x^3}$$ (not required)

2. $$f(x) = 2\sin(x^2) + x - 2, \qquad 0 \leqslant x < 2\pi$$

(a) Show that $f(x) = 0$ has a root α between $x = 0.75$ and $x = 0.85$

(2)

The equation $f(x) = 0$ can be written as $x = \left[\arcsin(1 - 0.5x)\right]^{\frac{1}{2}}$.

(b) Use the iterative formula

$$x_{n+1} = \left[\arcsin(1 - 0.5x_n)\right]^{\frac{1}{2}}, \quad x_0 = 0.8$$

to find the values of x_1, x_2 and x_3, giving your answers to 5 decimal places.

(3)

(c) Show that $\alpha = 0.80157$ is correct to 5 decimal places.

(3)

a) $f(0.75) = -0.18339\ldots < 0$ Change of
$f(0.85) = 0.1725\ldots > 0$ sign

$\Rightarrow 0.75 < \alpha < 0.85$

b) $x_0 = 0.8$
$x_1 = 0.80219$
$x_2 = 0.80133$
$x_3 = 0.80167$

c) $f(0.801575) = 8 \times 10^{-6} > 0$
$f(0.801565) = -2.7 \times 10^{-5} < 0$

change of sign $\Rightarrow \alpha \in (0.801565, 0.801575)$

$\therefore \alpha = 0.80157$ (5dp)

3.

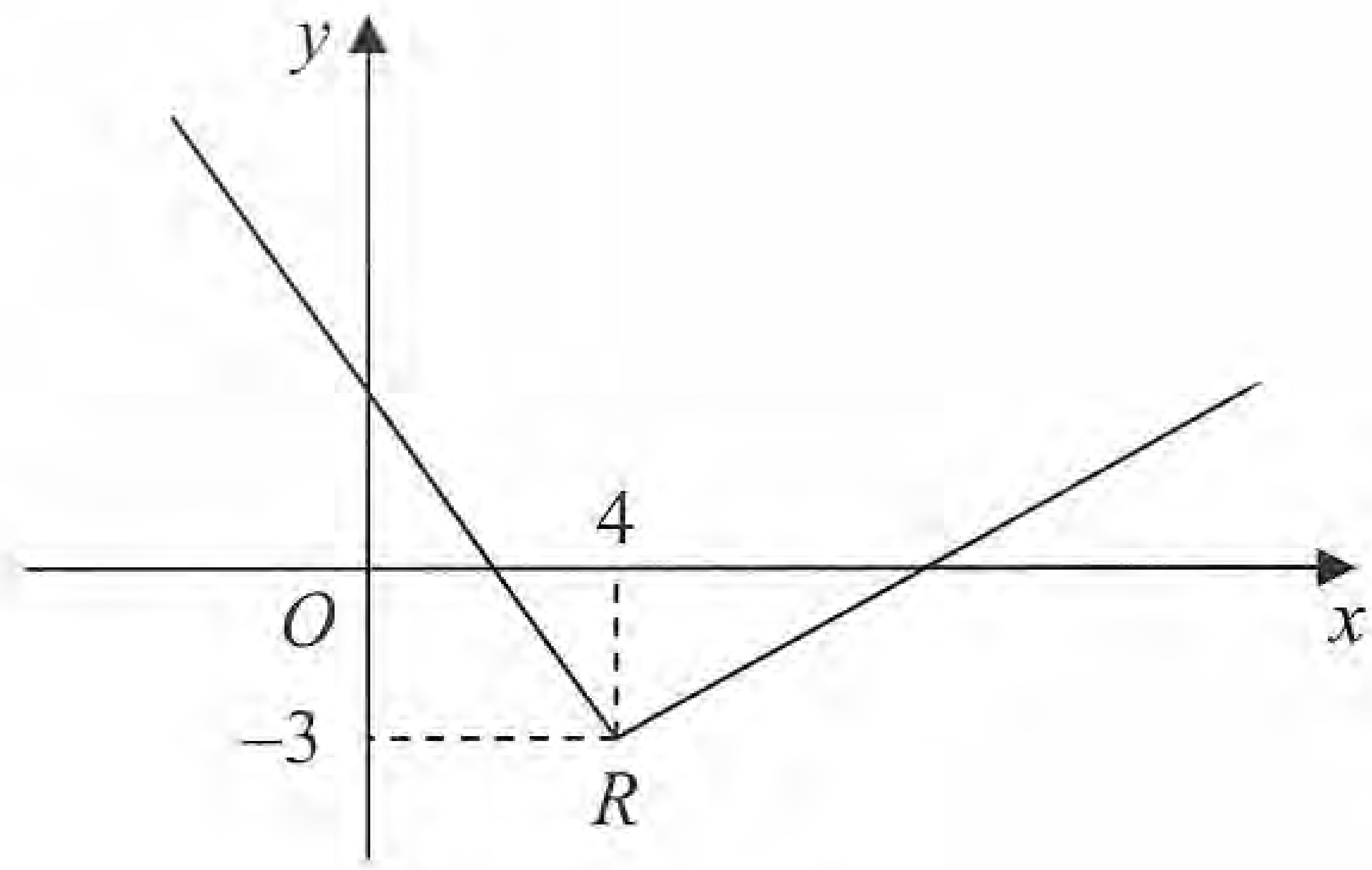


Figure 1

Figure 1 shows part of the graph of $y = \mathrm{f}(x)$, $x \in \mathbb{R}$.

The graph consists of two line segments that meet at the point $R(4, -3)$, as shown in Figure 1.

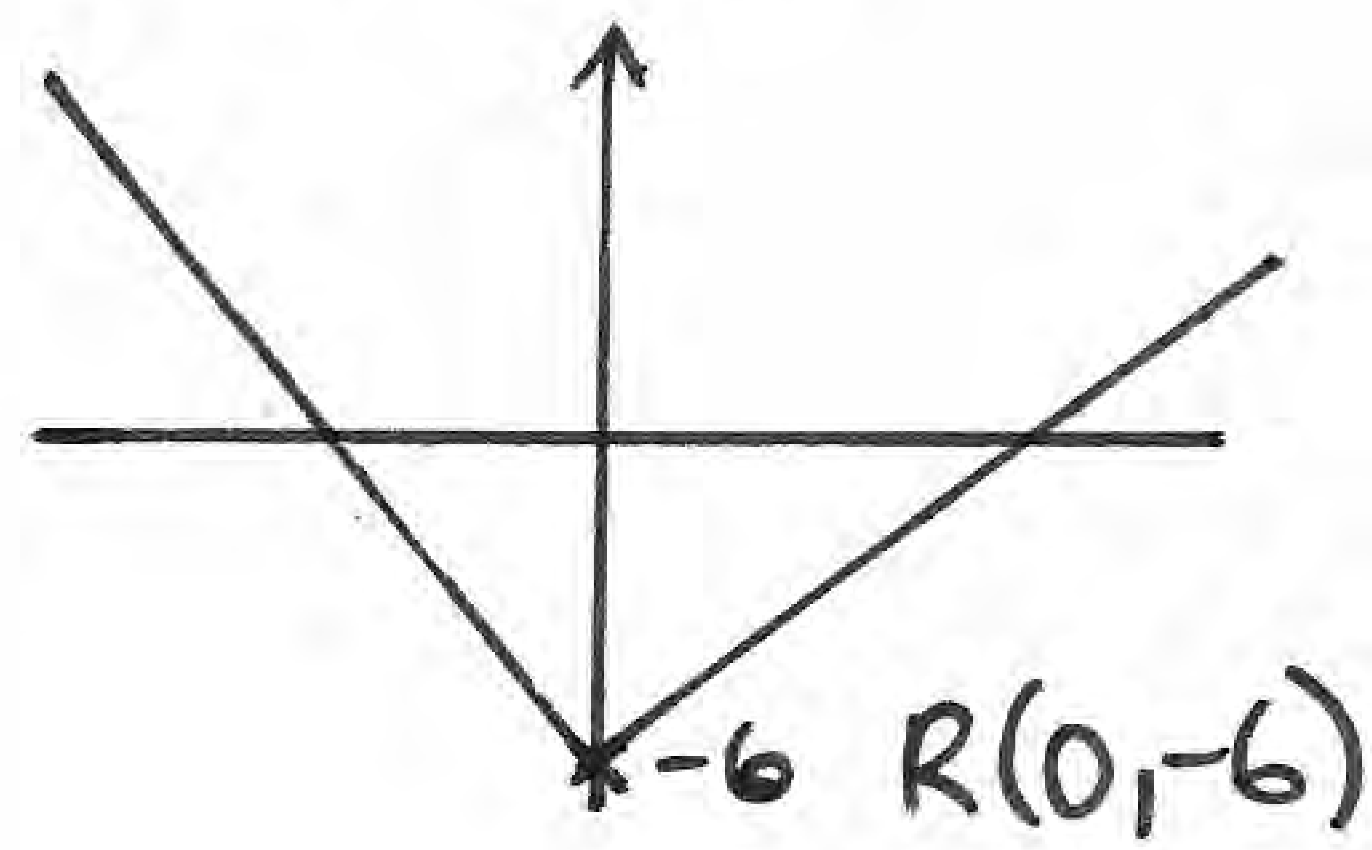
Sketch, on separate diagrams, the graphs of

(a) $y = 2\mathrm{f}(x+4)$, **(3)**

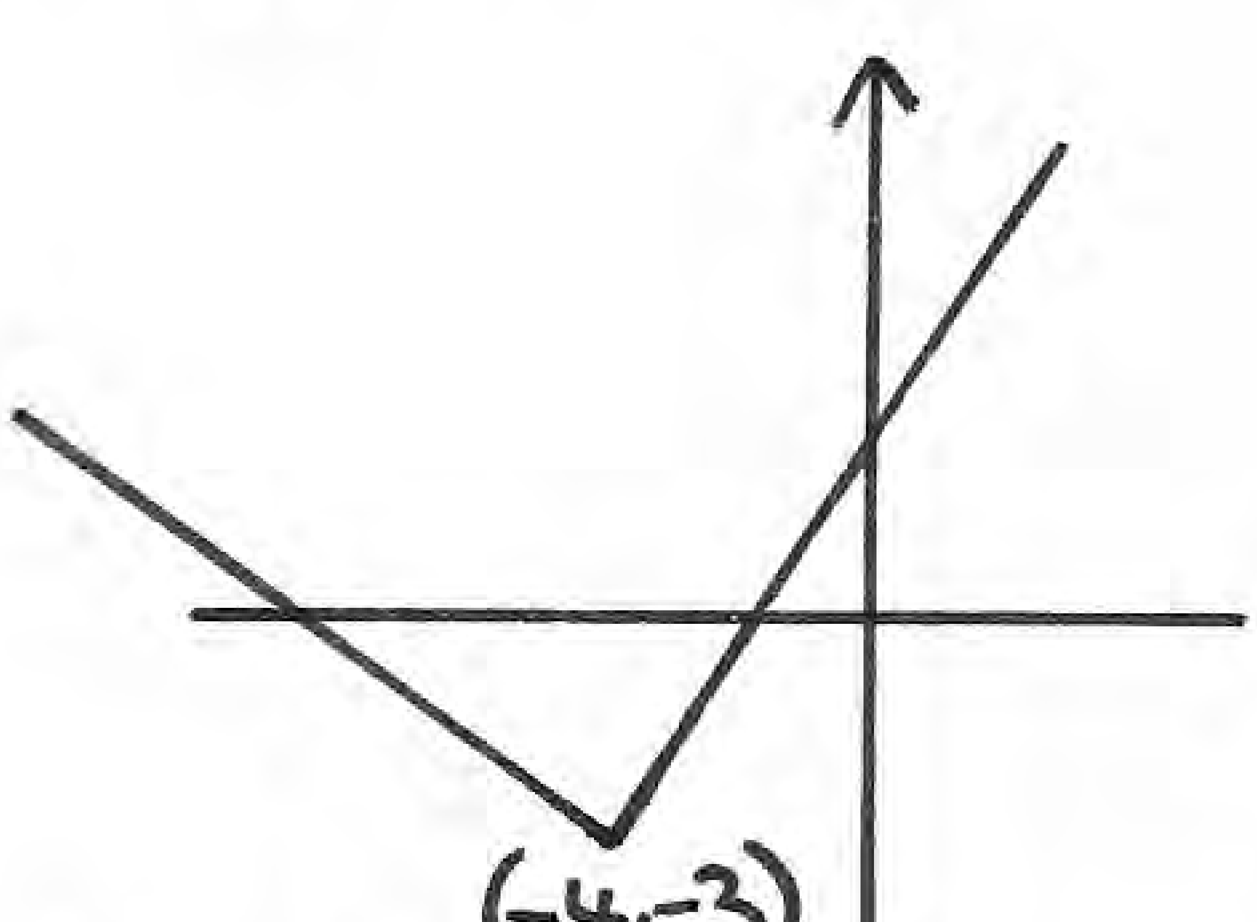
(b) $y = |\mathrm{f}(-x)|$. **(3)**

On each diagram, show the coordinates of the point corresponding to R.

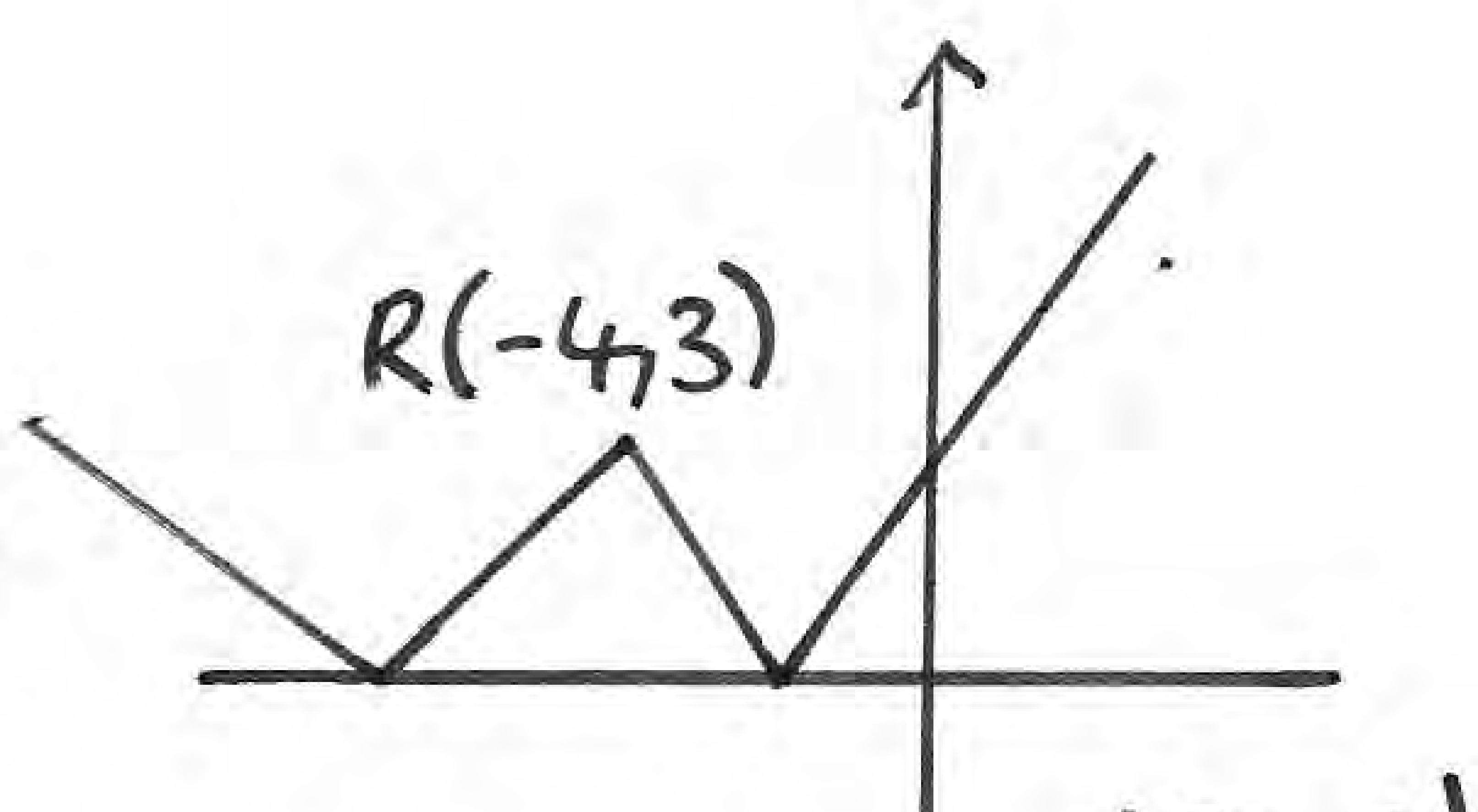
a) $y = 2f(x+4)$ ↑↓ $\xleftarrow{4}$



b) $y = |f(-x)|$



$\Rightarrow$



$f(-x)$ flip

$|f(-x)|$

4. The function f is defined by

$$f : x \mapsto 4 - \ln(x+2), \quad x \in \mathbb{R},\ x \geqslant -1$$

(a) Find $f^{-1}(x)$.

(3)

(b) Find the domain of f^{-1}.

(1)

The function g is defined by

$$g : x \mapsto e^{x^2} - 2, \quad x \in \mathbb{R}$$

(c) Find $fg(x)$, giving your answer in its simplest form.

(3)

(d) Find the range of fg.

(1)

a) $y = 4 - \ln(x+2) \Rightarrow x = 4 - \ln(y+2)$

$\Rightarrow \ln(y+2) = 4 - x \Rightarrow y+2 = e^{4-x} \Rightarrow y = -2 + e^{4-x} = f^{-1}(x)$

b)
$f(-1) = 4 - \ln(1) = 4$
$f(0) = 4 - \ln 2 < 4$ $\Rightarrow$ range $y \leq 4$ for $f(x)$

$\therefore$ domain $x \leq 4$ for $f^{-1}(x)$

c) $fg(x) = f(e^{x^2} - 2) = 4 - \ln(e^{x^2} - 2 + 2)$

$= 4 - \cancel{\ln}(e^{x^2}) = 4 - x^2$

d)

$x \longrightarrow g \longrightarrow f \longrightarrow fg(x)$

$x \in \mathbb{R}$ $\quad \geqslant -1$

$\Rightarrow$ range $y \leq 4$

5. The mass, m grams, of a leaf t days after it has been picked from a tree is given by

$$m = p\mathrm{e}^{-kt}$$

where k and p are positive constants.

When the leaf is picked from the tree, its mass is 7.5 grams and 4 days later its mass is 2.5 grams.

(a) Write down the value of p.

(1)

(b) Show that $k = \frac{1}{4}\ln 3$.

(4)

(c) Find the value of t when $\frac{\mathrm{d}m}{\mathrm{d}t} = -0.6\ln 3$.

(6)

a) $t=0 \quad m=7.5 \quad \Rightarrow 7.5 = p$

b) $t=4 \quad m=2.5 \Rightarrow 2.5 = 7.5e^{-4k}$

$\Rightarrow \frac{1}{3} = e^{-4k} \Rightarrow \ln\left(\frac{1}{3}\right) = -4k \Rightarrow -\ln 3 = -4k$

$\Rightarrow k = \frac{1}{4}\ln 3$ #

c) $m = 7.5e^{\left(-\frac{1}{4}\ln 3\right)t}$

$\frac{dm}{dt} = 7.5\left(-\frac{1}{4}\ln 3\right)e^{\left(-\frac{1}{4}\ln 3\right)t} = -0.6\ln 3$

$\Rightarrow \cancel{-}\frac{15}{8}\cancel{\ln 3}\, e^{\left(-\frac{1}{4}\ln 3\right)t} = \cancel{-}0.6\cancel{\ln 3}$

$\Rightarrow e^{\left(-\frac{1}{4}\ln 3\right)t} = \frac{8}{25} \Rightarrow \left(-\frac{1}{4}\ln 3\right)t = \ln\left(\frac{8}{25}\right)$

$\Rightarrow t = \frac{-4\ln\left(\frac{8}{25}\right)}{\ln 3} \Rightarrow t = \underline{4.15}\ (3sf)$

6. (a) Prove that

$$\frac{1}{\sin 2\theta} - \frac{\cos 2\theta}{\sin 2\theta} = \tan\theta, \quad \theta \neq 90n°, \ n \in \mathbb{Z}$$

(4)

(b) Hence, or otherwise,

(i) show that $\tan 15° = 2 - \sqrt{3}$,

(3)

(ii) solve, for $0 < x < 360°$,

$$\operatorname{cosec} 4x - \cot 4x = 1$$

(5)

a) $\frac{1 - \cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} = \frac{\cancel{2}\sin^{\cancel{2}}\theta}{\cancel{2}\cancel{\sin\theta}\cos\theta}$

$= \frac{\sin\theta}{\cos\theta} = \tan\theta$ #

b) $\tan 15 = \frac{1}{\sin 30} - \frac{\cos 30}{\sin 30} = \frac{1}{\frac{1}{2}} - \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \underline{2-\sqrt{3}}$

#

c) $\operatorname{cosec} 4x - \cot 4x = 1$

$\Rightarrow \frac{1}{\sin 4x} - \frac{\cos 4x}{\sin 4x} = 1 \Rightarrow \underline{\theta = 2x} \Rightarrow \tan 2x = 1$

$\Rightarrow 2x = \tan^{-1}(1) = 45, 225, 405, 585, \ldots$

$x = \underline{22.5}, \underline{112.5}, \underline{202.5}, \underline{292.5}°$

7. $$f(x) = \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{x^2-9}, \qquad x \neq \pm 3,\ x \neq -\frac{1}{2}$$

(a) Show that

$$f(x) = \frac{5}{(2x+1)(x+3)}$$

(5)

The curve C has equation $y = f(x)$. The point $P\left(-1, -\frac{5}{2}\right)$ lies on C.

(b) Find an equation of the normal to C at P.

(8)

a) $f(x) = \frac{4x-5}{(2x+1)(x-3)} - \frac{2x}{(x+3)(x-3)}$ $\times(x+3)$ $\times(2x+1)$

$= \frac{(4x-5)(x+3) - 2x(2x+1)}{(2x+1)(x-3)(x+3)}$

$= \frac{4x^2+7x-15-4x^2-2x}{(2x+1)(x-3)(x+3)} = \frac{5x-15}{(2x+1)(x-3)(x+3)}$

$= \frac{5\cancel{(x-3)}}{(2x+1)\cancel{(x-3)}(x+3)} = \frac{5}{(2x+1)(x+3)}$ #

b) $f(x) = 5(2x^2+7x+3)^{-1}$

$f'(x) = -5(2x^2+7x+3)^{-2} \times (4x+7)$

$f'(x) = \frac{-5(4x+7)}{(2x^2+7x+3)^2}$ $x=-1$ $m_t = \frac{-15}{4}$

$m_n = \frac{4}{15}$

$\therefore\ y + \frac{5}{2} = \frac{4}{15}(x+1)$

8. (a) Express $2\cos 3x - 3\sin 3x$ in the form $R\cos(3x+\alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. Give your answers to 3 significant figures.

(4)

$$f(x) = e^{2x}\cos 3x$$

(b) Show that $f'(x)$ can be written in the form

$$f'(x) = Re^{2x}\cos(3x+\alpha)$$

where R and α are the constants found in part (a).

(5)

(c) Hence, or otherwise, find the smallest positive value of x for which the curve with equation $y = f(x)$ has a turning point.

(3)

$R\cos(3x+\alpha) = R\cancel{\cos 3x}\cos\alpha - R\cancel{\sin 3x}\sin\alpha$

$2\cancel{\cos 3x} \quad - 3\cancel{\sin 3x}$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{2} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{2}\right) = 0.983$ (3sf)

$R^2 = 3^2 + 2^2 = 13 \Rightarrow R = 3.61$

$\sqrt{13}\cos(3x + 0.983)$

b) $u = e^{2x}$ $\quad v = \cos 3x$ $\qquad vu' + uv'$

$u' = 2e^{2x}$ $\quad v' = -3\sin 3x$

$f'(x) = 2e^{2x}\cos 3x - 3e^{2x}\sin 3x$

$= e^{2x}(2\cos 3x - 3\sin 3x)$

$= e^{2x}\left(\sqrt{13}\cos(3x + 0.983)\right)$

c) $f(x)$ has TP when $f'(x) = 0$

$$e^{2x}\sqrt{13}\left(\cos(3x + 0.983)\right) = 0$$

$$\Rightarrow \cos(3x + 0.983\ldots) = 0$$

$$\Rightarrow 3x + 0.983\ldots = \cancel{-\frac{\pi}{2}}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$3x = \frac{\pi}{2} - 0.983\ldots$$

$$x = \frac{\frac{\pi}{2} - 0.983\ldots}{3} = \underline{0.196} \text{ (3sf)}$$