C4 S15

1. (a) Find the binomial expansion of

$$(4+5x)^{\frac{1}{2}}, \quad |x| < \frac{4}{5}$$

in ascending powers of x, up to and including the term in x^2. Give each coefficient in its simplest form.

(5)

(b) Find the exact value of $(4+5x)^{\frac{1}{2}}$ when $x = \frac{1}{10}$

Give your answer in the form $k\sqrt{2}$, where k is a constant to be determined.

(1)

(c) Substitute $x = \frac{1}{10}$ into your binomial expansion from part (a) and hence find an approximate value for $\sqrt{2}$

Give your answer in the form $\frac{p}{q}$ where p and q are integers.

(2)

a) $4^{\frac{1}{2}}\left(1+\frac{5}{4}x\right)^{\frac{1}{2}} = 2\left(1+\frac{5}{4}x\right)^{\frac{1}{2}}$

$= 2\left[1+\left(\frac{1}{2}\right)\left(\frac{5}{4}x\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(\frac{5}{4}x\right)^2\right] = 2 + \frac{5}{4}x + -\frac{25}{64}x^2$

b) $\left(4+\frac{5}{10}\right)^{\frac{1}{2}} = \left(\frac{9}{2}\right)^{\frac{1}{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \quad \therefore k = \frac{3}{2}$

c) $x = \frac{1}{10} \Rightarrow \frac{3}{2}\sqrt{2} \simeq 2 + \frac{5}{4}\left(\frac{1}{10}\right) - \frac{25}{64}\left(\frac{1}{10}\right)^2 = \frac{543}{256}$

$\times\frac{2}{3}$ $\quad \therefore \sqrt{2} \simeq \frac{181}{128}$

2. The curve C has equation

$$x^2 - 3xy - 4y^2 + 64 = 0$$

(a) Find $\dfrac{dy}{dx}$ in terms of x and y.

(5)

(b) Find the coordinates of the points on C where $\dfrac{dy}{dx} = 0$

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(6)

a) $\frac{d}{dx}(x^2) - \frac{d}{dx}(3xy) - \frac{d}{dx}(4y^2) + \frac{d}{dx}(64) = 0$

$\Rightarrow 2x - 3y - 3x\frac{dy}{dx} - 8y\frac{dy}{dx} = 0$

$\Rightarrow (3x+8y)\frac{dy}{dx} = 2x - 3y \quad \therefore \frac{dy}{dx} = \frac{2x-3y}{3x+8y}$

b) $\frac{dy}{dx} = 0 \Rightarrow 2x - 3y = 0 \Rightarrow 2x = 3y \quad \therefore y = \frac{2}{3}x$

$\Rightarrow x^2 - \cancel{3}x\left(\frac{2}{\cancel{3}}x\right) - 4\left(\frac{2}{3}x\right)^2 + 64 = 0$

$\Rightarrow x^2 - 2x^2 - \frac{16}{9}x^2 + 64 = 0$

$(\times 9) \quad 9x^2 - 18x^2 - 16x^2 + 576 = 0$

$\Rightarrow 25x^2 = 576 \Rightarrow x^2 = \frac{576}{25} \Rightarrow x = \pm\frac{24}{5}$

$y = \frac{2}{3}x \quad \therefore y = \pm\frac{48}{15} = \pm\frac{16}{5} \qquad C\left(\frac{24}{5}, \frac{16}{5}\right); \left(\frac{-24}{5}, \frac{-16}{5}\right)$

3.

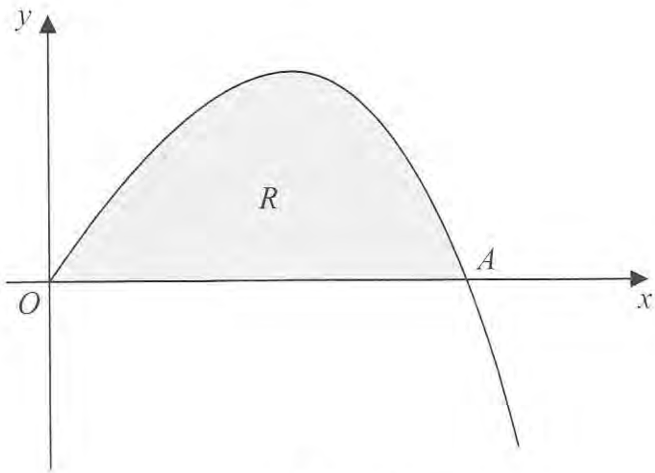


Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = 4x - xe^{\frac{1}{2}x}$, $x \geqslant 0$

The curve meets the x-axis at the origin O and cuts the x-axis at the point A.

(a) Find, in terms of $\ln 2$, the x coordinate of the point A.

(2)

(b) Find

$$\int xe^{\frac{1}{2}x}\,dx$$

(3)

The finite region R, shown shaded in Figure 1, is bounded by the x-axis and the curve with equation

$$y = 4x - xe^{\frac{1}{2}x}, \quad x \geqslant 0$$

(c) Find, by integration, the exact value for the area of R.
Give your answer in terms of $\ln 2$

(3)

a) $y=0 \Rightarrow 4\cancel{x} = \cancel{x}e^{\frac{1}{2}x} \Rightarrow \ln 4 = \frac{1}{2}x \quad \therefore x = 2\ln 4$

$\therefore x = 2\ln 2^2 = \underline{4\ln 2}$

b) $\int xe^{\frac{1}{2}x}dx \qquad \int uv' = uv - \int u'v \qquad u = x \quad v = 2e^{\frac{1}{2}x}$

$u' = 1 \quad v' = e^{\frac{1}{2}x}$

$= 2xe^{\frac{1}{2}x} - \int 2e^{\frac{1}{2}x}dx$

$= 2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x} + c \qquad \left(2(x-2)e^{\frac{1}{2}x} + c\right)$

c) $R = \int_0^{4\ln 2} 4x - xe^{\frac{1}{2}x}\,dx = \left[2x^2 - 2xe^{\frac{1}{2}x} + 4e^{\frac{1}{2}x}\right]_0^{4\ln 2}$

$x = 2\ln 4$
$\Rightarrow \frac{1}{2}x = \ln 4$

$= \left(32(\ln 2)^2 - 8\ln 2\, e^{\ln 4} + 4e^{\ln 4}\right] - (0 - 0 + 4)$

$= 32(\ln 2)^2 - 32\ln 2 + 16 - 4$

$= \underline{32(\ln 2)^2 - 32\ln 2 + 12}$

4. With respect to a fixed origin O, the lines l_1 and l_2 are given by the equations

$$l_1: \mathbf{r} = \begin{pmatrix} 5 \\ -3 \\ p \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \quad l_2: \mathbf{r} = \begin{pmatrix} 8 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$$

where λ and μ are scalar parameters and p is a constant.

The lines l_1 and l_2 intersect at the point A.

(a) Find the coordinates of A.

(2)

(b) Find the value of the constant p.

(3)

(c) Find the acute angle between l_1 and l_2, giving your answer in degrees to 2 decimal places.

(3)

The point B lies on l_2 where $\mu = 1$

(d) Find the shortest distance from the point B to the line l_1, giving your answer to 3 significant figures.

(3)

a) $r_1 = r_2 \Rightarrow \begin{pmatrix} 5 \\ -3+\lambda \\ p-3\lambda \end{pmatrix} = \begin{pmatrix} 8+3\mu \\ 5+4\mu \\ -2-5\mu \end{pmatrix}$ (i) $5 = 8+3\mu \Rightarrow \mu = -1$

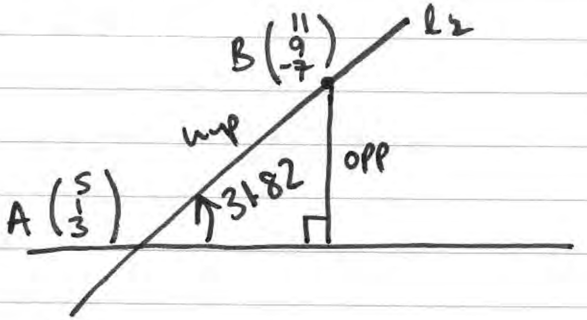
$\therefore A(8-3, 5-4, -2+5) \quad \therefore A(5,1,3)$

b) (j) $-3+\lambda = 5+4(-1) \Rightarrow -3+\lambda = 1 \quad \therefore \lambda = 4$

(k) $p-3(4) = -2-5(-1) \Rightarrow p-12 = -2+5 \quad \therefore p = 15$

c) $\cos\theta = \left| \dfrac{\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}}{\left|\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}\right| \left|\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}\right|} \right| \Rightarrow \cos\theta = \dfrac{19}{\sqrt{10}\sqrt{50}} \quad \therefore \theta = 31.82^\circ$

d) $B\begin{pmatrix} 8+3 \\ 5+4 \\ -2-5 \end{pmatrix} \quad B\begin{pmatrix} 11 \\ 9 \\ -7 \end{pmatrix} \quad B(11,9,-7)$ on l_2.



$\vec{AB} = \begin{pmatrix} 11 \\ 9 \\ -7 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ -10 \end{pmatrix}$

$\therefore |\vec{AB}| = \sqrt{6^2+8^2+10^2}$

$= \sqrt{200}$

$\therefore$ Shortest distance $= \sqrt{200} \times \sin 31.82$

$= 7.46$

5. A curve C has parametric equations

$$x = 4t + 3, \quad y = 4t + 8 + \frac{5}{2t}, \quad t \neq 0$$

(a) Find the value of $\frac{dy}{dx}$ at the point on C where $t = 2$, giving your answer as a fraction in its simplest form.

(3)

(b) Show that the cartesian equation of the curve C can be written in the form

$$y = \frac{x^2 + ax + b}{x - 3}, \quad x \neq 3$$

where a and b are integers to be determined.

(3)

a) $\frac{dy}{dt} = 4 - \frac{5}{2t^2}$ $\qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4 - \frac{5}{2t^2}}{4}$

$\frac{dx}{dt} = 4$

$\therefore \frac{dy}{dx} = \frac{8t^2 - 5}{8t^2}$

When $t = 2$,

$\frac{dy}{dx} = \frac{8(2)^2 - 5}{8(2)^2}$

$= 27/32$

b) $y = 4t + 8 + \frac{10}{4t}$ $\qquad 4t = x - 3$

$\therefore y = (x-3) + 8 + \frac{10}{x-3} = x + 5 + \frac{10}{x-3}$

$\therefore y = \frac{(x+5)(x-3)+10}{x-3} = \frac{x^2+2x-15+10}{x-3}$

$\therefore y = \frac{x^2+2x-5}{x-3}$

6.

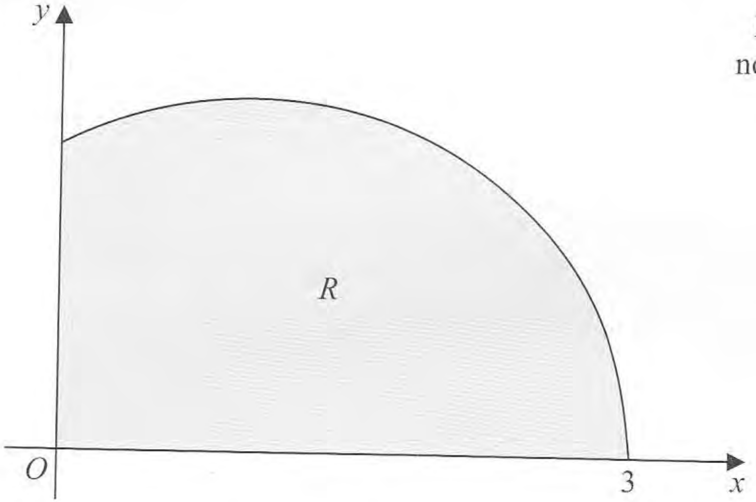


Diagram not to scale

Figure 2

Figure 2 shows a sketch of the curve with equation $y = \sqrt{(3 - x)(x + 1)}$, $0 \leqslant x \leqslant 3$

The finite region R, shown shaded in Figure 2, is bounded by the curve, the x-axis, and the y-axis.

(a) Use the substitution $x = 1 + 2\sin\theta$ to show that

$$\int_0^3 \sqrt{(3 - x)(x + 1)}\,\mathrm{d}x = k\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta$$

where k is a constant to be determined.

(5)

(b) Hence find, by integration, the exact area of R.

(3)

a) $\displaystyle\int_0^3 \sqrt{(3-x)(x+1)}\,dx$

$x = 1 + 2\sin\theta$

$\dfrac{dx}{d\theta} = 2\cos\theta \qquad dx = 2\cos\theta\,d\theta$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4(1-\sin\theta)(1+\sin\theta)} \times 2\cos\theta\,d\theta$$

$3 - x = 2 - 2\sin\theta = 2(1-\sin\theta)$

$x + 1 = 2 + 2\sin\theta = 2(1+\sin\theta)$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta} \times \cos\theta\,d\theta$$

$x = 3 \qquad 1 + 2\sin\theta = 3$

$2\sin\theta = 2$

$\sin\theta = 1$

$\theta = \frac{\pi}{2}$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\cos^2\theta} \times \cos\theta\,d\theta$$

$x = 0 \qquad 1 + 2\sin\theta = 0$

$2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

$\theta = -\frac{\pi}{6}$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

b)

$\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \frac{1}{2}\cos 2\theta + \frac{1}{2} = \cos^2\theta$

$$\Rightarrow 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos 2\theta + 1\,d\theta = 2\left[\tfrac{1}{2}\sin 2\theta + \theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2 \times \tfrac{1}{2} \times \left[\sin 2\theta + 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} = \left[\sin 2\theta + 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= (0 + \pi) - \left(-\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

7. (a) Express $\dfrac{2}{P(P-2)}$ in partial fractions.

(3)

A team of biologists is studying a population of a particular species of animal.

The population is modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P(P-2)\cos 2t, \quad t \geqslant 0$$

where P is the population in thousands, and t is the time measured in years since the start of the study.

Given that $P = 3$ when $t = 0$,

(b) solve this differential equation to show that

$$P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}$$

(7)

(c) find the time taken for the population to reach 4000 for the first time.
Give your answer in years to 3 significant figures.

(3)

a) $\dfrac{2}{P(P-2)} = \dfrac{A}{P} + \dfrac{B}{P-2} \Rightarrow 2 = A(P-2) + BP$

$P=0 \Rightarrow -2A \equiv 2 \therefore A=-1$ $\quad$ $P=2 \Rightarrow 2 = 2B \therefore B=1$

$= \dfrac{1}{P-2} - \dfrac{1}{P}$

b) $\displaystyle\int \frac{2}{P(P-2)}\,dP = \int \cos 2t\,dt$

$\displaystyle\int \frac{1}{P-2} - \frac{1}{P}\,dP = \frac{1}{2}\sin 2t + C$

$\Rightarrow \ln(P-2) - \ln P = \ln\left(\dfrac{P-2}{P}\right) = \dfrac{1}{2}\sin 2t + C$

$P=3, t=0 \Rightarrow \ln\left(\frac{1}{3}\right) = \frac{1}{2}\cancel{\sin 0} + c \quad \therefore c = -\ln 3$

$\Rightarrow \ln\left(\frac{P-2}{P}\right) + \ln 3 = \frac{1}{2}\sin 2t$

$\Rightarrow \frac{1}{2}\sin 2t = \ln\left(\frac{3P-6}{P}\right) \Rightarrow \frac{3P-6}{P} = e^{\frac{1}{2}\sin 2t}$

$\Rightarrow 3P - 6 = Pe^{\frac{1}{2}\sin 2t} \Rightarrow 3P - Pe^{\frac{1}{2}\sin 2t} = 6$

$\Rightarrow P\left(3 - e^{\frac{1}{2}\sin 2t}\right) = 6 \quad \therefore P = \frac{6}{3 - e^{\frac{1}{2}\sin 2t}}$

c) $\frac{6}{3-e^{\frac{1}{2}\sin 2t}} = 4000 \Rightarrow 3 - e^{\frac{1}{2}\sin 2t} = \frac{3}{2000}$

$\therefore e^{\frac{1}{2}\sin 2t} = 2.9985$

$\therefore \sin 2t = 2\ln(2.9985)$

(READ THE Qu)

c) $\frac{6}{3-e^{\frac{1}{2}\sin 2t}} = 4 \Rightarrow 3 - e^{\frac{1}{2}\sin 2t} = \frac{3}{2}$

$\therefore e^{\frac{1}{2}\sin 2t} = \frac{3}{2} \quad \therefore \sin 2t = 2\ln\left(\frac{3}{2}\right)$

$\therefore 2t = 0.9457\ldots$

$\therefore t = 0.473$

2

8.

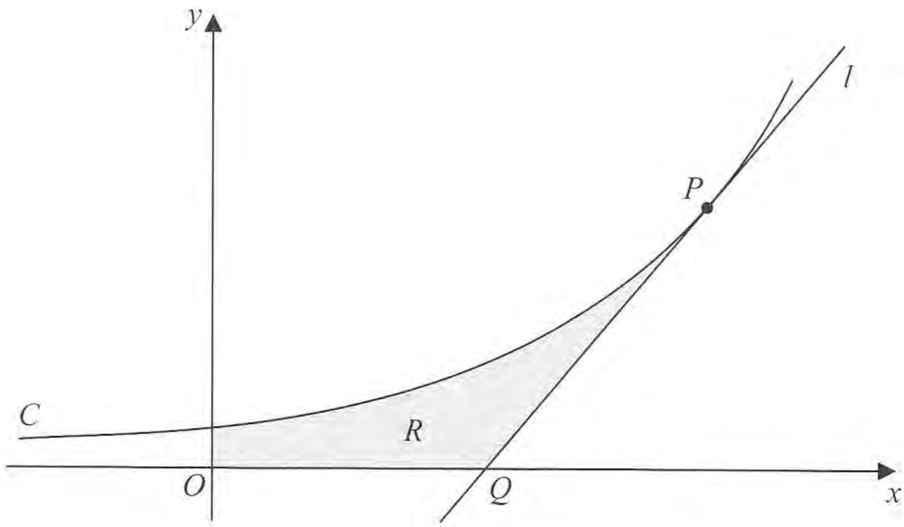


Diagram not to scale

Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = 3^x$$

The point P lies on C and has coordinates $(2, 9)$.

The line l is a tangent to C at P. The line l cuts the x-axis at the point Q.

(a) Find the exact value of the x coordinate of Q.

(4)

The finite region R, shown shaded in Figure 3, is bounded by the curve C, the x-axis, the y-axis and the line l. This region R is rotated through 360° about the x-axis.

(b) Use integration to find the exact value of the volume of the solid generated.

Give your answer in the form $\frac{p}{q}$ where p and q are exact constants.

[*You may assume the formula* $V = \frac{1}{3}\pi r^2 h$ *for the volume of a cone.*]

(6)

$y = 3^x \Rightarrow \frac{dy}{dx} = 3^x \ln 3$ when $x = 2$ M_t at $P = 9\ln 3$

$y - 9 = 9\ln 3(x-2)$ at Q $y = 0$

$-9 = 9\ln 3(x-2) \Rightarrow x - 2 = \frac{-9}{9\ln 3} \Rightarrow x = 2 - \frac{1}{\ln 3}$

$x = \frac{(2\ln 3) - 1}{\ln 3}$

b) $\pi \int_0^2 y^2\,dx$ $-$  Cone $r = 9$

$2 - \left(2 - \frac{1}{\ln 3}\right)$

$= \frac{1}{\ln 3}$

$= \pi \int_0^2 (3^x)^2\,dx - \frac{1}{3}\pi (9)^2 \times \frac{1}{\ln 3}$

$= \pi \int_0^2 3^{2x}\,dx - \frac{81\pi}{3\ln 3}$

$= \pi \left[\frac{3^{2x}}{2\ln 3} \right]_0^2 - \frac{81\pi}{3\ln 3}$

$= \pi \left[\frac{81}{2\ln 3} - \frac{1}{2\ln 3} \right] - \frac{81\pi}{3\ln 3}$

$= \frac{80\pi}{2\ln 3} - \frac{81\pi}{3\ln 3}$

$= \frac{40\pi}{\ln 3} - \frac{27\pi}{\ln 3} = \frac{13\pi}{\ln 3}$

$\int 3^{2x}\,dx$

let $u = 2x$ $\frac{du}{dx} = 2$

$dx = \frac{1}{2}du$

$\frac{1}{2}\int 3^u\,du$

$= \frac{1}{2}\frac{3^u}{\ln 3} = \frac{3^{2x}}{2\ln 3}$